POISSON IDEALS IN CLUSTER ALGEBRAS AND THE SPECTRA OF QUANTIZED COORDINATE RINGS

SEBASTIAN ZWICKNAGL

Abstract. We describe the Poisson ideals and attached symplectic geometry of a cluster algebra with compatible Poisson structure. We apply these results to determine the spectrum of a quantum cluster algebra. As an application, we describe the topology on the spectra of quantized coordinate rings such as quantum matrices and $\mathcal{O}_q(GL_n)$.

Contents

1. Introduction

Since cluster algebra structures were introduced by Fomin and Zelevinsky in 2000 (see [14]) they have found many applications from representation theory to mathematical physics to integrable systems. As the original motivation was to study canonical bases in quantum groups , they are intimately related to their classical limits–standard Poisson structures on Lie groups. Gekhtman, Shapiro and Vainsthein investigated these connections in a series of papers starting with [19] and culminating, for now, in the recent book [20]. However, the quantum counterpart, quantum cluster algebras, as introduced by Berenstein and Zelevinsky in [5], remained elusive. However, recently it could be shown that a number of quantized coordinate rings have, indeed, the structure of quantum cluster algebras, for example certain Grassmannians and partial flag varieties by Grabowski and Launois ([28] and [27]) and some subalgebras of nilpotent quantized universal enveloping algebras by Lampe [37], [38]. Most recently, Geiß-Leclerc and Schröer obtained quantum cluster algebra structures on quantized coordinate rings closely related to Schubert cells and partial flag varieties, including quantum matrices.

Such quantum coordinate rings have been an active area of research for the past twenty years. While the representation theory of quantized enveloping algebras is rather well-known, there is also quite a lot of interesting results regarding the representations, resp. the ideal theory of their duals, quantized coordinate rings $\mathcal{O}_q(G)$ (see e.g. the books by Joseph [33] and Brown and Goodearl [6, Part II]). The natural question to ask is to describe the prime and primitive spectra and their respective Zariski-type topologies. If one views a universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ as a quantization of the algebra $S(\mathfrak{g}^*)$ with the Kirillov-Kostant-Souriau Poisson structure, then the philosophy of the orbit method or geometric quantization (see e.g. [36]), suggests the following conjecture which has been open since the 1990s.

Conjecture 1.1. *(see e.g.* [6, Section II.10] *and* [21]*) Let G be a semisimple algebraic group with the standard Poisson-Lie structure, and $\mathcal{O}_q(G)$, the corresponding quantum group. Then the classical limit induces a homeomorphism from the set of primitive ideals of $\mathcal{O}_q(G)$ with the Zariski-type topology to the space of symplectic leaves on G.*

Indeed, this conjecture is part of a more general picture for quantized coordinate rings, as outlined in Goodearl's survey [21]. In the present paper, we investigate the ideal theory of quantum cluster algebras and cluster algebras to analyze the spectrum of many quantized coordinate rings, the most interesting result of which is the following theorem.

Theorem 1.2. *Let $G = GL_n(\mathbb{C}), SL_n(\mathbb{C})$. The classical limit induces a homeomorphism between the primitive spectrum of $\mathcal{O}_q(G)$ and the symplectic leaves of G with the standard Poisson Lie structure.*

In order to prove this result we first describe the Poisson spectra of a cluster algebra. A cluster algebra with a compatible Poisson structure which we will call a *Poisson cluster algebra* is given by a triple $(\mathbf{x}, B, \Lambda)$ where $\mathbf{x} = (x_1, \ldots, x_n) \subset \mathbb{C}(x_1, \ldots, x_n)$, the field of fractions in n indeterminates, is an extended cluster, B a skew-symmetrizable integer $m \times n$-matrix, the *exchange matrix*, and Λ a skew-symmetric $n \times n$-matrix satisfying a compatibility criterion (2.4) which we call the

Poisson matrix of the cluster. Indeed, the Poisson bracket on the cluster algebra $\mathfrak{A} \subset \mathbb{C}(x_1, \ldots, x_n)$ is defined by

$$\{x_i, x_j\} = \lambda_{ij} x_i x_j \ .$$

The cluster algebra is defined recursively by generating more and more cluster variables in the following way. For each variable x_i, $1 \leq i \leq m$ we define new cluster variables y_i as

$$y_i = x_i^{-1} \left(\prod_{k=1}^{n} x_k^{b_{ik}^+} + \prod_{k=1}^{n} x_k^{-b_{ik}^-} \right) \ ,$$

where $r^+ = \max(r, 0)$ and $r^- = \min(r, 0)$. This process is called mutation. The functions $x_{n+1}, \ldots, x_m$ remain frozen, and are referred to as *coefficients*. Let Y be the set $Y = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. The following theorem describes the Poisson spectrum $P.spec(\mathfrak{A})$ of a Poisson cluster algebra $\mathfrak{A}$ with compatible Poisson bracket, i.e. the spectrum of ideals which are both Poisson and prime.

Theorem 1.3. *Let $\mathfrak{A}$ be a Poisson cluster algebra, $(\mathbf{x}, B, \Lambda)$ and Y as above. Then, $P.spec(\mathfrak{A})$ can be stratified as*

$$P.spec(\mathfrak{A}) = \bigsqcup_{S \subset Y} P.spec_S(\mathfrak{A}) \ .$$

Moreover, if $P.spec_S(\mathfrak{A})$ is nonempty, then it is homeomorphic to the spectrum of an Poisson torus, resp. its Poisson center, a Laurent polynomial ring.

The theorem follows rather straightforwardly from the author's classification of torus invariant Poisson prime ideals in [50]. We make use of the combinatorial data B and Λ to determine for which $S \subset Y$ when $P.spec_S(\mathfrak{A}) \neq \emptyset$, as well as to compute the rank of the Laurent polynomial ring. Additionally, we can describe the inclusion relations between ideals.

Quantum cluster algebras $\mathfrak{A}_q$ are defined by triples $(\mathbf{x}, B, \Lambda)$ as well. Indeed the corresponding Poisson cluster algebra $\mathfrak{A}$ is the classical limit of $\mathfrak{A}_q$ Using the classical limit, we obtain the following result which is analogous to Theorem 1.3.

Theorem 1.4. *Let $\mathfrak{A}_q$ be a quantum cluster algebra, $(\mathbf{x}, B, \Lambda)$ and Y as above. Then, $P.spec(\mathfrak{A})$ can be stratified as*

$$P.spec(\mathfrak{A}) = \bigsqcup_{S \subset Y} P.spec_S(\mathfrak{A}) \ .$$

Moreover, $P.spec_S(\mathfrak{A})$ is homeomorphic to the spectrum of a quantum torus, resp. its center, a Laurent polynomial ring.

We now obtain the analogue of the orbit method for quantum cluster algebras.

Theorem 1.5. *Let $\mathfrak{A}_q$ be a quantum cluster algebra and $\mathfrak{A}$, a Poisson cluster algebra its classical limit. Then $spec(\mathfrak{A}_q)$ and $P.spec(\mathfrak{A})$ are homeomorphic.*

Now, in the case when $\mathfrak{A}$ is finitely generated and the algebra of functions on an affine complex variety $\mathfrak{A} = \mathbb{C}[X]$, then we show that the symplectic leaves are algebraic varieties. Now, we apply these results to the quantum cluster algebra structures on the quantum $n \times n$ matrices, resp. their localization at the quantum determinant $\mathcal{O}_q(G)$, and obtain Theorem 1.2.

We will now briefly recall some of the history and motivation for this paper, and explain the methods used in it. Recall that primitive ideals in non-commutative algebras are very important for the representation theory since they are the annihilators of irreducible modules. Originally, Kirillov used the orbit method to describe the representation theory of a nilpotent Lie group G, by establishing a homeomorphisms between the irreducible representations and the coadjoint orbits of G on $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g} = Lie(G)$ of G (see [35]). It was then discovered independently by Kirillov, Kostant and Souriau that the coadjoint orbits coincide with the symplectic leaves of the so-called Kirillov-Kostant-Souriau Poisson bracket on $S(\mathfrak{g}^*)$ whose "quantization" is the universal enveloping algebra $U(\mathfrak{g})$.

With the emergence of quantum groups it was natural to ask for corresponding relations in a more general setting. The first major accomplishment were the following results of Soibelman and Vaksman: If K is a compact Lie group, then the irreducible representations of the generic quantized coordinate rings of K are in a bijective correspondence with symplectic leaves on K with the Poisson bracket derived from the quantization (see [42], [43] and [44]). A similar, combinatorial bijection was later established for semisimple $\mathfrak{g}$ by Hodges and Levasseur for SL_n ([29] and [30]) and in the general case by Hodges, Levasseur and Toro in [31]. Using different methods, Joseph accomplished the same result in [32] where he also discusses the topology on the primitive ideals, does not, however, link it to symplectic leaves. As such Conjecture 1.1 has remained an open conjecture for more than a decade (see e.g. [6, Problem II.10.11 and Conjecture II.10.12]. here has been a lot of recent progress studying prime or primitive ideals in quantum Schubert cells employing a variety of methods. On the one hand there is the approach by Cauchon and Mériaux [9] using the method of deleting derivations to study the torus invariant prime ideals, later used by Bell, Casteels, Launois and Nguyen to study properties of the torus strata in a series of papers [1], [2] and [3]. More extensive results were obtained by Yakimov in his papers [45], [46] and [48], where he not only classifies the torus invariant prime ideals, and their inclusion relations, but also relates them to the Poisson geometry of flag varieties studied in the papers [7] and [26] by Brown, Goodearl and Yakimov. In his recent work, Yakimov [49] uses these results to establish a natural bijection between symplectic leaves and primitive ideals. for a large number of quantized coordinate rings, including the quantized function algebras of semisimple algebraic groups and the algebras $\mathfrak{a}_q(w)$ studied in Section 5.1.

An excellent reference would be Goodearl's paper [21]. He also outlines a broader program of what the orbit method should mean for quantized coordinate rings and we propose here that it may be possible to prove this correspondence in more generality using the ring theory of cluster algebras and upper cluster algebras, respectively their quantum analogues. A key ingredient in this study is the understanding of stratifications like the ones in Theorems 1.3 and 1.4, which are defined by the sets of torus invariant prime ideals. The general theory is known as Goodearl-Letzter Stratification Theory for Noetherian rings (see the discussion in Appendix B and [6, Chapter II.2]). Indeed Theorem 1.4 is an analogue of a Goodearl-Letzter stratifications for quantum cluster algebras, while Theorem 1.3 is a Poisson version.

As Goodearl shows in [21], the spectra of $\mathfrak{A}$ and $\mathfrak{A}_q$ are homeomorphic if we can show that the bijective map of spectra induced by the classical limit, as well as its inverse, preserve inclusions. This follows from our explicit descriptions (Theorem

1.3 and Theorem 1.4) . The key notion for our argumentation which can be defined using cluster algebras are the *defining clusters* of prime ideals (see Section 3.2.1). Notice however we do not require the cluster algebras or quantum cluster algebras to be Noetherian.

In general,the orbit method has been very successful in understanding the representation theory and geometry of Lie groups and algebras. In the classical instances, it is believed (see e.g. Kirillov [36]) that it works because of properties of the logarithm, in particular the Campbell-Baker-Hausdorff-formula. The quantizations of cluster algebras and cluster varieties are closely related to dilogarithm functions (see e.g. Fock and Goncharov's papers [10], [11] and Keller's preprint [34]). Therefore, we conjecture that there should be a dilogarithmic analogue of the orbit method (see Problem 6.2).

The paper is organized as follows. Section 2 reviews definitions and facts about cluster algebras and compatible Poisson structures. In Section 3 we discuss the Poisson spectrum of a cluster algebra, while the following Section 4 discusses the spectrum of a quantum cluster algebra. We apply the results to quantized coordinate rings in Section 5. Some open questions will be discussed in Section 6. An appendix contains material on specializations of quantum algebras including the classical limit and Goodearl-Letzter stratification theory. The reader will be accompanied through the first three sections by our standard example, the Grassmannian $G(2,5)$ of two-dimensional subspaces in a five-dimensional vectorspace.

Acknowledgements 1.6. *The author would like to thank Jan Schröer and for helpful conversations, and Ken Goodearl, Jan Grabowski, Stephane Launois as well as Milen Yakimov for very valuable comments on a draft of this paper.*

2. Cluster Algebras

2.1. **Cluster algebras.** In this section we will review the definitions and some basic results on cluster algebras, or more precisely, on cluster algebras of geometric type. We will, however, introduce cluster algebras over a field k and not, as usual over $\mathbb{Z}$, but we can consider our cluster algebras as obtained from the standard version by tensoring with k over $\mathbb{Z}$. Let k be a field of characteristic zero and denote by $\mathfrak{F} = k(x_1, \ldots, x_n)$ the field of fractions over k in n indeterminates. In order to obtain a cluster algebra we define an algorithm to produce further transcendence bases of $\mathfrak{F}$ corresponding to choices of skew-symmetrizable $m \times n$-matrices B, where $m \leq n$. More precisely, recall that a $m \times n$-integer matrix B is called skew-symmetrizable if there exists a $n \times n$- diagonal matrix with positive integer entries D such that $D \cdot B$ is skew-symmetric. We call the tuple $(x_1, \ldots, x_n, B)$ the *initial seed* of the cluster algebra and $\mathbf{x} = (x_1, \ldots x_m)$ a cluster. We will now construct more clusters, $\mathbf{y} = (y_1, \ldots, y_n)$, which are transcendence bases of $\mathfrak{F}$, and seeds $(\mathbf{y}, \tilde{B})$ in the following way.

Define for each real number r the numbers $r^+ = \max(r, 0)$ and $r^- = \min(r, 0)$. Given a skew-symmetrizable integer $m \times n$-matrix B, we define for each $1 \leq i \leq m$ the *exchange polynomial*

$$P_i = \prod_{k=1}^{n} x_k^{b_{ik}^+} + \prod_{k=1}^{n} x_k^{-b_{ik}^-} \; . \tag{2.1}$$

We can now define the new cluster variable x_i' via the equation

$$x_i x_i' = P_i \ .$$

This allows us to refer to the matrix B as the *exchange matrix* of the cluster $(x_1, \ldots, x_n)$.

It is easily verified that $(x_1, x_2, \ldots, \hat{x}_i, x_i', x_{i+1}, \ldots, x_n)$ is another transcendence basis of $\mathfrak{F}$. We now define the new exchange matrix $B_i = B'$, associated to the new cluster

$$\mathbf{x}_i = (x_1, x_2, \ldots, \hat{x}_i, x_i', x_{i+1}, \ldots, x_n)$$

by defining the coefficients b_{ij} as follows:

- $b'_{ij} = -b_{ij}$ if $j \leq n$ and $i = k$ or $j = k$,
- $b'_{ij} = b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}$ if $j \leq n$ and $i \neq k$ and $j \neq k$,
- $b'_{ij} = b_{ij}$ otherwise.

We call this algorithm *matrix mutation.* Note that B_i is again skew-symmetrizable (see e.g. [14]). The process of obtaining a new seed is called *cluster mutation.*

Definition 2.1. *The cluster algebra $\mathfrak{A} \subset \mathfrak{F}$ corresponding to an initial seed $(x_1, \ldots, x_n, B)$ is the subalgebra of $\mathfrak{F}$, generated by the elements of all the clusters (transcendence bases of $\mathfrak{F}$) obtained from the initial cluster through a finite number of cluster mutations. We call the elements of the clusters the cluster variables.*

Remark 2.2. *If $m < n$, then there will be some cluster variables $x_{m+1}, \ldots, x_n$ which will never be mutated. We will refer to these variables as coefficients, even though in the literature they are sometime called "frozen".*

We have the following fact, motivating the definition of cluster algebras in the study of total positivity phenomena and canonical bases.

Proposition 2.3. [14, Section 3]*(Laurent phenomenon) Let $\mathfrak{A}$ be a cluster algebra with initial cluster $(x_1, \ldots, x_n)$. Any cluster variable x can be expressed uniquely as a Laurent polynomial with integer coefficients in $x_1, \ldots, x_n$.*

Moreover, it has been conjectured, and proven in a number of cases see (see .e.g. [40] and then [12],[13]) that the coefficients are positive; i.e., that the Laurent polynomials are subtraction free.

2.2. **Our Standard Example.** In this section we introduce what will become the standard example for our quantum and classical cluster algebras–the coordinate ring $\mathbb{C}[G(2,5)]$ of the Grassmannian $G(2,5)$ the variety of two-dimensional subspaces of $\mathbb{C}^5$. We define it as the subalgebra of the $\mathbb{C}[Mat_{2,5}]$, the functions on 2×5-matrices, generated by the 2×2-minors,

$$\Delta_{ij} = x_{i1}x_{j2} - x_{i2}x_{j1} \ .$$

It is well-known that the minors are subject to the Plücker relations

$$\Delta_{ik}\Delta_{j\ell} = \Delta_{ij}\Delta_{k\ell} + \Delta_{i\ell}\Delta_{jk} \ , \tag{2.2}$$

for $1 \leq i < j < k < \ell \leq 5$. The algebra $\mathbb{C}[G(2,5)]$ has a natural cluster algebra structure with initial seed $(\mathbf{x}, B)$ where the cluster variables are minors (see e.g.[41]):

$$\boldsymbol{\Delta_{13}},\ \boldsymbol{\Delta_{14}}, \quad \Delta_{12},\ \Delta_{23},\ \Delta_{34},\ \Delta_{45},\ \Delta_{15}\ .$$

The latter five functions are coefficients and the exchange matrix has the following shape:

$$B = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix} . \tag{2.3}$$

The exchange relations are therefore:

$$\Delta_{13}\ y_1 = \Delta_{14}\Delta_{23} + \Delta_{12}\Delta_{34}\ ,$$

$$\Delta_{14}\ y_2 = \Delta_{34}\Delta_{15} + \Delta_{13}\Delta_{45}\ .$$

We observe from (2.2) that $y_1 = \Delta_{24}$ and $y_2 = \Delta_{35}$. Indeed, the minors Δ_{ij} form the set of cluster variables. The cluster algebra is a cluster algebra of finite type A_2 in the classification of [15].

2.3. **Compatible Pairs and Their Mutation.** In this section we introduce compatible pairs and their mutation, following their definition in [5]. Let $m \leq n$. Consider a pair consisting of a skew-symmetrizable $m \times n$-integer matrix B with rows labeled by $[1, m]$ and columns labeled by an n-element s$[1, m]$ and a skew-symmetrizable $n \times n$-integer matrix Λ with rows and columns labeled by $[1, n]$.

Definition 2.4. *Let B and Λ be as above. We say that the pair (B, Λ) is compatible if for all $j \in \mathbf{ex}$ and $i \in [1, m]$ one has:*

$$\sum_{k=1}^{m} b_{kj}\lambda_{ki} = \delta_{i,j} d_j\ ,$$

for some positive integers d_j $(j \in [1, m])$.

This means that $D = B \cdot \Lambda$ is a $m \times n$ matrix where the only non-zero entries are positive integers on the diagonal of the principal $m \times m$-submatrix.

Let (B, Λ) be a compatible pair and let $k \in [1, m]$. We define for $\varepsilon \in \{+1, -1\}$ a $n \times n$ matrix $E_{k,\varepsilon}$ via

$$E_{k,\varepsilon} = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ -1 & \text{if } i = j = k \\ max(0, -\varepsilon_{ik}) & \text{if } i \neq j = k \end{cases}\ , \tag{2.4}$$

and a $m \times m$ matrix $F_{k,\varepsilon}$ via

$$F_{k,\varepsilon} = \begin{cases} \delta_{ij} & \text{if } i \neq k \\ -1 & \text{if } i = j = k \\ max(0, \varepsilon_{kj}) & \text{if } i = k \neq j \end{cases}\ .$$

We define a new pair (B_k, Λ_k) as

$$B_k = F_{k,\varepsilon} B E_{k,\varepsilon}\ , \quad \Lambda_k = E_{k,\varepsilon}^T \Lambda E_{k,\varepsilon}\ , \tag{2.5}$$

where X^T denotes the transpose of X. We have the following fact.

Proposition 2.5. [5, Prop. 3.4] *The pair (B_k, Λ_k) is compatible. Moreover, Λ_k is independent of the choice of the sign ε.*

2.4. **Poisson structures.** In this section we recall the definition and properties of Poisson algebras, and discuss the relationship between symplectic leaves and symplectic cores. We will follow mostly the exposition and notation in [21, Section 3–7] where the reader will also find proofs for most of the facts.

Definition 2.6. *Let k be a field of charactieristic 0. A Poisson algebra is a pair $(A, \{\cdot,\cdot\})$ of a commutative k-algebra A and a bilinear map $\{\cdot,\cdot\} : A \otimes A \to A$ satisfying for all $a, b, c \in A$:*

1. *skew-symmetry: $\{a, b\} = -\{b, a\}$*
2. *Jacobi identity: $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$,*
3. *Leibniz rule: $a\{b, c\} = \{a, b\}c + b\{a, c\}$.*

If there is no room for confusion we will refer to a Poisson algebra as A instead of $(A, \{\cdot,\cdot\})$.

Gekhtman, Shapiro and Vainshtein showed in [19] that one can associate Poisson structures to cluster algebras. We have collected some basic definitions and facts about Poisson algebras in Appendix to [50] and will refer to them in this section when necessary. Let $\mathfrak{A} \subset k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subset \mathfrak{F}$ be a cluster algebra. A Poisson structure $\{\cdot,\cdot\}$ on $k[x_1, \dots, x_n]$ is called log-canonical if for all $1 \leq i, j \leq n$, $\{x_i, x_j\} = \lambda_{ij} x_i x_j$ for $\lambda_{ij} \in \mathbb{C}$. We call $\Lambda = \{\omega_{ij}\}_{i,j=1}^n$ the coefficient matrix of the Poisson structure. We say that a Poisson structure is compatible with $\mathfrak{A}$ if it is log-canonical with respect to each cluster $(y_1, \dots, y_n)$; i.e. log canonical on $k[y_1, \dots, y_n]$.

The following result is an adapted version of [19, Theorem 1.4].

Theorem 2.7. [19, Theorem 1.4] *and* [20] *Let B be an irreducible skew-symmetrizable integer $m \times n$-matrix of rank m and suppose that $D = diag(d_1, \dots, d_m)$ is a diagonal matrix with positive integer entries such that $D \cdot B$ is skew-symmetric . A Poisson structure is compatible with the cluster algebra structure if and only if its coefficient matrix Λ satisfies $B \cdot \Lambda = D$; i.e. (B, Λ) is a compatible pair.*

Remark 2.8. *We will refer to the cluster algebra and compatible Poisson structure $(\mathbf{x}, B, \Lambda)$ as a Poisson cluster algebra.*

2.5. **Toric Actions.** Let $\mathfrak{A}$ be a cluster algebra and let $\mathbf{x} = (x_1, \dots, x_n)$ be a cluster. Following [19, Section 2.3] we define for each element $w = (w_1, w_2, \dots, w_n) \in \mathbb{Z}^n$ a *local toric action* of k^* via maps : $(x_1, \dots, x_n) \mapsto (\alpha^{w_1} x_1, \dots \alpha^{w_n} x_n)$ for all $\alpha \in k^*$. Assume now that we have chosen integer weights $w_{\mathbf{x}} = (w_1, w_2, \dots, w_n)$ for each cluster $\mathbf{x}$. We say that a local toric action is compatible with the cluster algebra structure, if the following diagram commutes for any two clusters $\mathbf{x} = (x_1, \dots, x_i, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, x_i, \dots, y_n)$:

$$\begin{tikzcd} k[\mathbf{x}] \arrow[r, "T"] \arrow[d, "\psi_{\mathbf{x},\alpha}"] & k[\mathbf{y}] \arrow[d, "\psi_{\mathbf{y},\alpha}"] \\ k[\mathbf{x}] \arrow[r, "T"] & k[\mathbf{y}] \end{tikzcd} ,$$

where T is a series of mutations transforming $\mathbf{x}$ to $\mathbf{y}$. Compatible local toric actions define *global toric actions* on the cluster algebra. We have the following fact.

Lemma 2.9. [19, Lemma 2.3] *Let Z denote the transformation matrix of the cluster algebra at the cluster $\mathbf{x}$. The local toric action at $\mathbf{x}$ defined by $w \in \mathbb{Z}^n$ can be*

extended to a global toric action if and only if $Z \cdot w = 0$. Moreover, if such an extension exists it is unique.

2.6. **Our running example.**

2.6.1. *The standard Poisson structure.* In the case of $\mathbb{C}[G(2,5)]$ we have the so called standard Poisson structure which is compatible with the cluster algebra structure. It is defined via

$$\{\Delta_{ij}, \Delta_{k\ell}\} = (sgn(i-k) + sgn(j-\ell))\, \Delta_{i\ell}, \Delta_{kj} \ .$$

We observe that the Poisson bracket in the cluster

$$\boldsymbol{\Delta_{13}},\ \boldsymbol{\Delta_{14}},\qquad \Delta_{12},\ \Delta_{23},\ \Delta_{34},\ \Delta_{45},\ \Delta_{15}$$

is given by the matrix

$$\Lambda = \begin{pmatrix} \mathbf{0} & \mathbf{-1} & 1 & -1 & -1 & -2 & -1 \\ \mathbf{1} & \mathbf{0} & 1 & 0 & -1 & -2 & -1 \\ -1 & -1 & 0 & -1 & -2 & -2 & -1 \\ 1 & 0 & 1 & 0 & -1 & -2 & 0 \\ 1 & 1 & 2 & 1 & 0 & -1 & 0 \\ 2 & 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix} . \tag{2.6}$$

It can be verified by direct computation that (B, Λ) is a compatible pair.

2.6.2. *The toric actions.* The torus actions on the cluster algebra are given by the usual torus actions on the Grassmannian, i.e. the action of $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^5$ on $M \in Mat_{2,5}$ via left-, resp. right-multiplication by diagonal matrices:

$$\begin{pmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{pmatrix} \cdot M \cdot \begin{pmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{pmatrix} .$$

The weights of these actions on the initial cluster are $(i = 1, 2)$:

$$wt(\ell_i) = (1,1,1,1,1,1,1),\ wt(r_1) = (1,1,1,0,0,0,1),\ wt(r_2) = (0,0,1,1,0,0,0)\ ,$$

$$wt(r_3) = (1,0,0,1,1,0,0),\ wt(r_4) = (0,1,0,0,1,1,0),\ wt(r_5) = (0,0,0,0,0,1,1)\ .$$

It is easy to verify that the weights span the kernel of B. However, notice that there is actually an action of a six-dimensional torus on the Grassmannian–indeed, not all global toric actions can be "observed" at every cluster.

3. The Poisson spectrum of a cluster algebra

In this section we will discuss the symplectic and Poisson geometry attached to a cluster algebra. First, we briefly recall some definitions and facts regarding Poisson ideals. A detailed discussion can be found in [21]. A *Poisson ideal* of $(A, \{\cdot,\cdot\})$ is an ideal I in A such that $\{a, h\} \in I$ for all $a \in A$ and $h \in I$. Recall that we can endow the set of prime ideals $spec(A)$ in a noetherian ring A with the *Zariski topology*. The closed sets are defined as $V(I) := \{P \in spec(A) : P \supset I\}$ for some ideal $I \in A$. Similarly, we have the Zariski topology on the set of maximal ideals $maxspec(X)$. Now consider the case of a Poisson algebra A. A Poisson prime ideal $P \subset A$ is a Poisson ideal such that if I and J are Poisson ideals and $I \cdot J \subset P$ then $I \subset P$ or

$J \subset P$. Note that if A is a noetherian algebra over a field of characteristic 0, then a Poisson prime ideal must be prime (see [21, Lemma 6.2]). We denote the set of Poisson prime ideals by $P.spec(A)$, the *Poisson spectrum*. It also comes equipped with the natural Zariski-type topology.

Given any ideal I, we call the maximal Poisson ideal $\mathfrak{P}(I)$ such that $\mathfrak{P}(I) \subset I$, the *Poisson core* of I. It is the sum of all Poisson ideals contained in I. Note that the Poisson core of a prime ideal is a Poisson prime ideal.

The Poisson core of a maximal ideal is called a *Poisson primitive ideal*. The Poisson primitive spectrum $P.prim(A)$ is the set of all Poisson primitive ideals. It is a subset of $P.spec(A)$ and we endow it with th relative topology. We obtain a continuous, surjective map

$$maxspec(A) \to P.prim(A) \ ,$$

and its fibres are called the *symplectic cores*. We obtain a stratification

$$maxspec(A) = \bigsqcup_{P \in P.prim(A)} \{\mathfrak{m} \in maxspec(A) : \mathbb{P}(\mathfrak{m}) = P\} \ .$$

Poisson and symplectic cores were originally introduced by Brown and Gordon in [8], and we refer the reader for a more detailed discussion of their properties to [21].

3.1. **Toric Poisson Ideals in Cluster Algebras.** In this section, we recall the results of the author on torus invariant Poisson prime ideals in cluster algebras in [50]. The main result is the following theorem.

Theorem 3.1. [50, Theorem 1.1] *Let $\mathfrak{A}$ be a Poisson cluster algebra. Then $\mathfrak{A}$ contains finitely many torus invariant Poisson prime ideals.*

More precisely, given a cluster $\mathbf{x} = (x_1, \dots, x_n)$ we are able to describe the structure of the clusters as follows. Denote, as above by y_i the cluster variables obtained from $\mathbf{x}$ by mutation in direction i. We denote $Y = \{x_1, \dots, x_n, y_1, \dots, y_m\}$. Recall that the algebra $\mathcal{L}_{\mathbf{x}} = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]/\langle x_i y_i = P_i : 1 \leq i \leq m\rangle$ is called the *lower bound* associated to the seed ([4]). We have the following fact.

Proposition 3.2. [50] *Let $\mathfrak{A}$ be a cluster algebra $\mathbf{x}$ a cluster, and x_i, y_i and Y as defined above. Suppose that the exchange polynomials are coprime (see [4]).*
(a) Let $\mathcal{I}_S \subset \mathcal{L}_{\mathbf{x}}$ be the ideal generated by $S \subset Y$. The ideal $\mathcal{I}_S$ is a toric Poisson prime ideal if

(1) *$P_i \in \mathcal{I}_S$ is equivalent to $x_i \in \mathcal{I}_S$ or $y_i \in \mathcal{I}_S$($\mathcal{I}_S$ is prime)*
(2) *$\{z, z'\} \in \mathcal{I}_S$ for all $z \in \mathcal{I}_S$ and $z' \in Y$ ($\mathcal{I}_S$ is Poisson)*

(b) If $\mathcal{I}_S$ is a Poisson prime ideal in $\mathcal{L}_{\mathbf{x}}$, then it induces a unique toric Poisson prime ideal in $\mathfrak{A}$ and the upper bound of $\mathfrak{A}$.

We call the subset of the power set of Y defining the torus invariant prime ideals PP_Y.

3.2. **The Poisson spectrum.** The following theorem collects the results from [50] with regards to the Poisson spectrum. If J_S is a toric Poisson prime ideal, denote by $P.spec_S(\mathfrak{A})$ the Poisson prime ideals $\mathcal{I}$ for which $(H : \mathcal{I}) = J$, that means $J = \bigcap_{h \in H} h(\mathcal{I})$.

Theorem 3.3. *Let $\mathfrak{A}$ be a cluster algebra, or a finitely generated upper cluster algebra. The Poisson spectrum is stratified as*

$$P.spec(\mathfrak{A}) = \bigsqcup_{S \in PP_Y} P.spec_Y(\mathfrak{A}) \ .$$

Moreover, $P.spec_S(\mathfrak{A})$ stratifies into torus orbits of Poisson prime ideals.

3.2.1. *Defining Clusters for Ideals.*

Definition 3.4. *Let $\mathfrak{A}$ be a Poisson cluster algebra and $\mathcal{I} \subset \mathfrak{A}$ a torus invariant Poisson prime ideals. A cluster $\mathbf{x}$ is called a defining cluster for $\mathfrak{A}$ if $\mathcal{I}$ is defined by $S \subset Y$ and $S \supset S' \in PP_Y$ implies that $(S' - S) \cap \mathbf{x} \neq \emptyset$.*

The definition is important because of the following fact. Denote by $tdeg$ the transcendence degree of an algebra.

Proposition 3.5. *Suppose $\mathbf{x}$ is a defining cluster for $\mathcal{I} \subset \mathfrak{A}$ with $\mathcal{I} \cap Y = S$. Then $|\{x\} - S| = tdeg(\mathfrak{A}/\mathcal{I})$.*

Proof. We, clearly, have $|\{x\} - S| \leq tdeg(\mathfrak{A}/\mathcal{I})$. Since $\mathbf{x}$ is defining, we conclude that if $x_i \in S$, then $y_i \in S$ as well for all $1 \leq i \leq m$. Consider the ideal J generated by the x_i, y_i for $x_i \in S$, $1 \leq i \leq m$ and the $x_j \in S$ for $m+1 \leq j \leq n$. It is contained in $\mathcal{I}$. But $tdeg(\mathfrak{A}/J) \leq |\{x\} - S|$ by the following fact.

Lemma 3.6. *Let Y, $\mathbf{x}$, S and J be as above. Then each element $z \in \mathfrak{A}/J$ has a representative $\tilde{z} \in \mathfrak{A}$ such that $\tilde{z} \in \mathbb{C}[\{x_j^{\pm 1} : x_j \notin S]$.*

Proof.
Recall that by definition, for each exchangeable index $i \in [1, n]$ we have $\mathfrak{A} \subset \mathbb{C}[x_1^{\pm 1} \ldots, x_{i-1}^{\pm 1}, x_i, y_i, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$. Now, suppose that $z \in \mathfrak{A}/J$ has no representative in $\tilde{z} \in \mathbb{C}_\Lambda[\{x_j^{\pm 1} : x - j \notin S]$. Choose any representative z' and $x_i \in S$ and express z' as a polynomial in $\mathbb{C}[x_1^{\pm 1} \ldots, x_{i-1}^{\pm 1}, x_i, y_i, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Recall the following fact which is adapted from [50]. Suppose that $u \in \mathfrak{A}$ is a Laurent polynomial in $\mathbb{C}_\Lambda[x_{k_1}^{\pm 1}, \ldots, x_{k_r}^{\pm 1}]$ for some set $\{k_1, \ldots, k_r\} \subset [1, n]$. We want to show that $u \in \mathbb{C}_\Lambda[x_{k_1}, y_{k_1}, x_{k_2}^{\pm 1}, \ldots, x_{k_r}^{\pm 1}]$. Suppose not. We have $u = f + y_{k_1} g$ where $f \in \mathbb{C}[x_{k_1}, x_{k_2}^{\pm 1}, \ldots, x_{k_r}^{\pm 1}]$ and $g \in \mathbb{C}[y, x_{k_2}^{\pm 1}, \ldots, x_{k_r}^{\pm 1}]$ with $g \neq 0$. We now obtain that $x_{k_1} y_{k_1} = x_w h + h'$ for some $w \notin \{k_1, \ldots, k_r\}$ and $h, h' \in \mathbb{C}[x_1, \ldots, x_n]$. If, however, we express u as a Laurent polynomial, we obtain a Laurent polynomial involving a term $x_{k_1}^{-1} x_w$. Hence, we obtain the desired contradiction. Let $S \cap \mathbf{x} = \{x_{i_1}, \ldots, x_{i_k}\}$. Inductively, we can now write $z' = x_{i_1} f_1 + y_{i_1} g_1 + \ldots + x_{i_r} f_r + y_{i_r} g_r + \tilde{z}$ where $f_{i_k} \in \mathbb{C}[x_j^{\pm 1}, x_{i_k} : j \notin \{i_1, \ldots, i_k\}]$, $g_{i_k} \in \mathbb{C}[x_j^{\pm 1}, y_{i_k} : j \notin \{i_1, \ldots, i_k\}]$ and $\tilde{z} \in \mathbb{C}[x_j^{\pm 1} : x_j \notin S]$. The element $\tilde{z}$ is the desired representative. The lemma is proved. □

The proposition is proved. □

Now, given a cluster $\mathbf{x}$ and $S \in PP_Y$ we construct a defining cluster for $\mathcal{I}_S$ in the following way.

Algorithm 3.7. [50, Algorithm 3.13] *(a) Start with $\mathbf{x}$ and choose, if possible, one i such that $x_i \in S$ and $y_i \notin S$. If there is no such i, then the algorithm terminates.*
(b) Consider the cluster $\mathbf{x}_i$ and the set S_i, defines for $\mathbf{x}_i$, just as S is defined for $\mathbf{x}$.
(c) Repeat Step (a) with $\mathbf{x} = \mathbf{x}_i$.

3.2.2. *Torus invariant Poisson prime ideals in* $\mathbb{C}[G(2,5)]$. The set Y we have to consider is

$$Y = \{\mathbf{\Delta_{13}}, \mathbf{\Delta_{24}},\ \mathbf{\Delta_{14}},\ \mathbf{\Delta_{35}}, \quad \Delta_{12},\ \Delta_{23},\ \Delta_{34},\ \Delta_{45},\ \Delta_{15}\}\ .$$

The torus orbits of symplectic leaves that have codimension one are generated by the coefficients $\Delta_{12},\ \Delta_{23},\ \Delta_{34},\ \Delta_{45},\ \Delta_{15}$. Notice that the cluster variables which are not coefficients do not generate Poisson ideals, as e.g.

$$\{\Delta_{13}, \Delta_{24}\} = 2\Delta_{14}\Delta_{23}\ .$$

Now, consider $S = \{\Delta_{12}, \Delta_{23}\} \subset Y$. $S \notin PP_Y$ and does not define a toric Poisson prime ideal, since an ideal that contains S but neither Δ_{13} nor Δ_{24} cannot be prime by the Plücker relation (2.2)

$$\Delta_{13}\Delta_{24} = \Delta_{14}\Delta_{23} + \Delta_{12}\Delta_{34} \in \mathcal{I}\ .$$

However, one easily verifies that $S_1 = \{\Delta_{12}, \Delta_{23}, \Delta_{13}\}$, $S_2 = \{\Delta_{12}, \Delta_{23}, \Delta_{24}\}$ and $S_3 = \{\Delta_{12}, \Delta_{23}, \Delta_{13}, \Delta_{24}\}$ define toric Poisson prime ideals.

3.2.3. *Defining Clusters.* Let $S = \{\mathbf{\Delta_{13}},\ \mathbf{\Delta_{14}},\ \Delta_{12},\ \Delta_{23},\ \Delta_{34},\ \Delta_{45},\ \Delta_{15}\} \subset Y$. Since $\Delta_{24}, \Delta_{35} \notin S$, we observe that the transcendence degree of the quotient $\mathbb{C}[G(2,5)]/\mathcal{I}_S$ is 2. We construct a defining cluster as follows. First, apply mutation to Δ_{13} and obtain the cluster

$$\mathbf{x}' = \{\mathbf{\Delta_{24}},\ \mathbf{\Delta_{14}},\ \Delta_{12},\ \Delta_{23},\ \Delta_{34},\ \Delta_{45},\ \Delta_{15}\}\ .$$

Next we apply mutation to $\mathbf{\Delta_{14}}$ and we end up with the defining cluster

$$\mathbf{x}'' = \{\mathbf{\Delta_{24}},\ \mathbf{\Delta_{25}},\ \Delta_{12},\ \Delta_{23},\ \Delta_{34},\ \Delta_{45},\ \Delta_{15}\}\ .$$

Notice, that $\Delta_{25} \notin Y$, but it is clear from the transcendence degree of the quotient that $\Delta_{25} \notin \mathcal{I}_S$, hence we have indeed found a defining cluster.

3.3. **Symplectic Leaves and Poisson Primitive Ideals.** Now, given a Poisson cluster algebra $\mathfrak{A}$, we will study the Poisson primitive ideals, and in the case when it is finitely generated, the symplectic leaves, .

Let $\mathbf{x}$ be a cluster, $S \in PP_Y$ and $\mathcal{I}_S$ the corresponding toric Poisson prime ideal. Let $\{x_{i_1}, \ldots, x_{i_k}\} = s \cap \mathbf{x}$ and denote $\mathbf{i} = \{i_1, \ldots i_k\}$. Denote by Λ the coefficient matrix of the Poisson bracket in the cluster $\mathbf{x}$ and by $\Lambda_{\mathbf{i}}$ the submatrix obtained by removing the rows and columns corresponding to the elements of $\mathbf{i}$. By Algorithm 3.7 we may assume that $\mathbf{x}$ is a defining cluster for $\mathcal{I}_S$.

Theorem 3.8. *(a) Let* $\mathfrak{A}$ *be a Poisson cluster algebra defined by* $(\mathbf{x}, B, \Lambda)$ *and suppose that* $\mathbf{x}$ *is defining for the toric Poisson prime ideal* $\mathcal{I}_S$*. Then* $P.spec_S(\mathfrak{A})$ *is homeomorphic to* $P.spec(H_{\Lambda_{\mathbf{i}}})$.

(b) The Poisson primitive ideals are the Poisson prime ideals which are maximal in their respective strata, corresponding to maximal ideals in a Laurent polynomial ring.

Proof. Define for each $\ell \in \mathbb{Z}_{\geq 0}$ the subalgebra $\mathfrak{A}^\ell \subset \mathfrak{A}$ generated by the cluster variables obtained from $\mathbf{x}$ by up to ℓ mutations. Here $\mathfrak{A}^0 = \mathbb{C}[\mathbf{x}]$ and $\mathfrak{A}_1$ is known as the lower bound introduced in [4]. Notice that $\mathfrak{A}$ is the union or direct limit of the $\mathfrak{A}^\ell$. We have the following fact.

Lemma 3.9. *The algebras* $\mathfrak{A}^\ell$ *for* $\ell \geq 0$ *are Poisson algebras.*

Proof. We obtain from Lemma 4.8 that $\mathfrak{A}^\ell$ is the classical limit of the algebra $\mathfrak{A}_q^\ell$ introduced in Section 4.4. Therefore, it must be a Poisson algebra. □

Since we assumed that all the exchange polynomials are coprime, we know that $\mathfrak{A}_1 \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_{i-1}^{\pm 1}, x_i, y_i, x_{i+1}^{\pm 1}, \ldots, x_j, y_j, x_{j+1}^{\pm 1} \ldots x_n^{\pm 1}]$(see [4]) and hence that $\mathfrak{A}_1$ is a Poisson algebra. The intersection of every torus invariant Poisson prime ideal $\mathcal{I}_S$ in $\mathfrak{A}$ with $\mathfrak{A}_1$ is also a torus invariant prime ideal, generated by S (see also [50]). Indeed, we showed in [50] that the torus invariant Poisson prime ideals in $\mathfrak{A}^\ell$ are also induced by the sets $S \in PP_Y$ We have the following fact.

Proposition 3.10. *Let $\mathfrak{A}$ be a Poisson cluster algebra defined by $(\mathbf{x}, B, \Lambda)$ and let $S \in PP_Y$. Then $P.spec_S(\mathfrak{A}^\ell)$ is homeomorphic to the Poisson spectrum of the Poisson torus $H_{\Lambda_{\mathbf{i}}}$. The Poisson primitive ideals form a torus orbit.*

Proof.

Since none of the variables x_j, $j \notin \mathbf{i}$ are contained in the ideals in $P.spec_S(\mathfrak{A}_1)$, $P.spec_S(\mathfrak{A}^1)$ is homeomorphic to the Poisson spectrum of the localization at the multiplicative set generated by the $\mathbb{Z}$-linear span x_j, $j \notin \mathbf{i}$. Indeed, this set is Poisson multiplicative, since $\{s, z\} \in S_{\mathbf{i}}$ if $s, z \in S_{\mathbf{i}}$. Employing Lemma 3.6, we obtain that the localized algebra is isomorphic to the Poisson torus $H_{\Lambda_{\mathbf{i}}}$.

The Poisson ideals in the Poisson torus H_{Ω_i} can be described explicitly. Let $c_1, \ldots c_t \in \mathbb{Z}^s$ be a basis for the kernel of $\Omega_{\mathbf{i}}$. Notice that the *Poisson center*, i.e. the subalgebra $\mathcal{Z} \subset H_{\Omega_i}$ for which $\{\mathcal{Z}, H_{\Omega_i}\} = 0$, is the algebra $\mathcal{Z} = \mathbb{C}[x^{\pm c_1}, \ldots x^{\pm c_t}]$. We have the following fact.

Lemma 3.11. [21, 9.6(b)] *The Poisson ideals of the Poisson torus $H_{\Lambda_{\mathbf{i}}}$ are induced by the ideals in $\mathbb{C}[x^{\pm c_1}, \ldots x^{\pm c_t}]$. The Poisson primitive ideals correspond to points in the torus $\mathbb{C}[x^{\pm c_1}, \ldots x^{\pm c_t}]$.*

The proposition is proved. □

Now, we return to the proof of the theorem. Suppose that there exists a proper Poisson prime ideal $\mathcal{I}_H$ in $H_{\Lambda_{\mathbf{i}}}$ which does not induce a proper Poisson prime ideal in $\mathfrak{A}$. Then, the ideal it defines must contain a unit a and its inverse a^{-1}. However, then there exists $\ell \in \mathbb{Z}_{\geq 0}$ such that a and a^{-1} lie in the ideal defined by $\mathcal{I}_H$ in $\mathfrak{A}^\ell$ which contradicts Proposition 3.10.

Suppose that there exists a Poisson prime ideal $\mathcal{I} \subset \mathfrak{A}$ which is not induced by a Poisson prime ideal in $H_{\Lambda i}$. Then there exists ℓ such that $\mathfrak{A}^\ell \cap \mathcal{I}$ is not induced by a Poisson prime ideal in $H_{\Lambda_{\mathbf{i}}}$. This, once again, contradicts Proposition 3.10. Theorem 3.8 (a) is proved.

We will now prove part (b). The assertion clearly holds for the algebras $\mathfrak{A}^\ell$ because they are affine complex algebras and the Poisson primitive ideals are induced by the maximal ideals in $\mathbb{C}[x^{\pm c_1}, \ldots x^{\pm c_t}]$. It can be also obtained from the Poisson Dixmier-Moeglin equivalence proven by Goodearl in [22]. The general case follows by an argument analogous to the one in the proof of part (a). Theorem 3.8 is proved. □

We now consider the special case when $\mathfrak{A}$ is finitely generated, hence the algebra of functions on a complex affine variety X.

Theorem 3.12. *Let $\mathfrak{A} = \mathbb{C}[X]$ be a finitely generated Poisson cluster algebra defined by $(\mathbf{x}, B, \Lambda)$ and suppose that $\mathbf{x}$ is defining for the toric Poisson prime ideal $\mathcal{I}_S$. Denote by $\mathfrak{V}(S)$ the zero locus of $\mathcal{I}_S$. Then $\mathfrak{V}(S)$ contains a unique open*

torus orbit of symplectic leaves. Each leaf is algebraic, and its dimension equals the rank of $\Lambda_{\mathbf{i}}$.

Proof.

We showed in [50] that $\mathfrak{V}(S)$ is a torus orbit of symplectic cores. It, therefore, suffices to show that one of the symplectic leaves is algebraic (hence a symplectic core) and compute its rank. First, consider a generic point $p \in \mathfrak{V}(S)$, that is a point on which none of the $x_j \notin S$ vanish. It is obvious that the dimension of the symplectic leave containing p has dimension $k_{\mathbf{i}} = rank(\Lambda_{\mathbf{i}})$. Recall additionally that every symplectic leave in $\mathfrak{V}(S)$ contains a generic point (see [50]). This implies that all the symplectic leaves are $k_{\mathbf{i}}$-dimensional. It now remains to observe that the symplectic core of a generic point has dimension at most $k_{\mathbf{i}}$. Let $\mathcal{I}_p$ be the maximal Poisson ideal which vanishes at p. Now, consider the localization at the multiplicative set defined by the $x_j \notin S$. We obtain the algebra $\mathbb{C}[x_{j_1}^{\pm 1}, \ldots x_{j_s}^{\pm 1}]$ with Poisson bracket given by $\Lambda_{\mathbf{i}}$.

Since p is generic, each of the functions x^{c_k} is defined at p. Consider the ideal $\tilde{\mathcal{I}}'_p$ in $\mathbb{C}[x_{j_1}^{\pm 1}, \ldots x_{j_s}^{\pm 1}]$ generated by $\{x^{c_1} - x^{c_1}(p), \ldots x^{c_t} - x^{c_1}(p)\}$ and the ideal $\mathcal{I}'_p$ it induces in $\mathfrak{A}$. It is a Poisson ideal which is contained in the Poisson core of p. However, this implies that the dimension of the symplectic core is less or equal to the rank of $\Lambda_{\mathbf{i}}$. Hence, the symplectic leaves and cores have the same dimension. The symplectic core is connected as a complex manifold, and since the symplectic leaves are analytically open sets, each core can contain only one symplectic leaf. This follows from the fact that one cannot decompose a connected k-dimensional manifold as a disjoint union of more than one k-dimensional manifold. The theorem is proved. □

With regards to the inclusion properties we refer to the discussion in Section 4.4.1. Briefly, if $\mathcal{I} \subset \mathcal{J}$ are two Poisson prime ideals and $\mathcal{I} \in P.spec_{J_S}(\mathfrak{A})$ and $\mathcal{J} \in P.spec_{J_{S'}}(\mathfrak{A})$, then $S \supset S'$. Hence, we can use Algorithm 3.7 to construct a cluster $\mathbf{x}'$ which is defining for both $\mathcal{I}$ and $\mathcal{J}$. Now, we can compare ideals by comparing inclusion relations inside the Poisson affine space $\mathbb{C}[x'_1, \ldots, x'_n]$.

3.4. **The Case of $\mathbb{C}[G(2,n)]$.** We continue our running example. First, we describe the symplectic structure of the open torus orbit in terms of the combinatorial data $(\mathbf{x}, B, \Lambda)$, introduced in Sections 2.2 and 2.6.

We have to compute a $\mathbb{Z}$-basis for the kernel of Λ. Since the rank of Λ is six, we obtain the basis-vector $w = (1,0,1,-1,-2,2,0)$. Hence, Poisson primitive ideals in the Poisson torus H_Λ are generated by the functions $(\Delta_{13}\Delta_{12}\Delta_{23}^{-1}\Delta_{34}^{-2}\Delta_{45}^2 - \alpha)$ where $\alpha \in \mathbb{C}^*$. The corresponding symplectic leaves $\mathcal{I}_\alpha$, $\alpha \in \mathbb{C}^*$ of the Grassmannian $G(2,5)$ are defined by the condition

$$\Delta_{13}\Delta_{12}\Delta_{45}^2 - \alpha\Delta_{23}\Delta_{34} = 0\ . \tag{3.1}$$

Now first consider the ideals in the stratum defined by $S = \{\Delta_{15}\}$. We observe that $w' = (1,0,1,-1,-2,2)$ spans the kernel of Λ_S. Hence, a Poisson prime ideal in this stratum lies in the closure of $\mathcal{I}_\alpha$, if and only if it contains $(\Delta_{13}\Delta_{12}\Delta_{45}^2 - \alpha\Delta_{23}\Delta_{34})$.

On the other hand, let us consider Poisson prime ideals in the stratum defined by $S = \{23, 34, 35\}$. We easily observe that none of the these ideals is contained in the closure of $\mathcal{I}_\alpha$, as $\Delta_{23} = 0$ and (3.1) imply that one of the functions Δ_{13}, Δ_{12} or Δ_{45} vanish.

4. Quantum Cluster Algebras and Their Spectra

In this section we recall the definition of a quantum algebra, introduced by Berenstein and Zelevinsky in [5] and determine its spectrum.

4.1. **Quantum Cluster Algebras.** Recall the definition of a compatible pair from Section 2.3.

We define, for each skew-symmetric $n \times n$-matrix Λ, the skew-polynomial ring $\mathbb{C}^t_\Lambda[x_1, \ldots, x_n]$ to be the $\mathbb{C}[t^{\pm 1}]$-algebra generated by $x_1, \ldots, x_n$ subject to the relations

$$x_i x_j = t^{\lambda_{ij}} x_j x_i \ .$$

Analogously, the quantum torus H^t_Λ is defined as the localization of $\mathbb{C}^t_\Lambda[x_1, \ldots, x_n]$ at the monoid generated by $x_1, \ldots, x_n$, which is an Ore set. The quantum torus is clearly contained in the skew-field of fractions $\mathcal{F}_\Lambda$ of $\mathbb{C}^t_\Lambda[x_1, \ldots, x_n]$, and the Laurent monomials define a lattice $L \subset H^t_\Lambda \subset \mathcal{F}_\Lambda$ isomorphic to $\mathbb{Z}^n$. Denote by $x^{1e_1, \ldots, e_n}$ the monomial $x_1^{e_1} \ldots x_n^{e_n}$.

We are now ready to define the notion of a toric frame in order to define the quantum cluster algebra.

Definition 4.1. *A toric frame in $\mathfrak{F}$ is a mapping $M : \mathbb{Z}^m \to \mathfrak{F} - \{0\}$ of the form*

$$M(c) = \phi(X^{\eta(c)}) \ ,$$

where ϕ is a $\mathbb{Q}(\frac{1}{2})$-algebra automorphism of $\mathfrak{F}$ and $\eta : \mathbb{Z}^m \to L$ an isomorphism of lattices.

Since a toric frame M is determined uniquely by the images of the standard basis vectors $\phi(X^{\eta(e_1)})$,..., $\phi(X^{\eta(e_n)})$ of $\mathbb{Z}^m$, we can associate to each toric frame a skew commutative $m \times m$-integer matrix Λ_M. We can now define the quantized version of a seed.

Definition 4.2. [5, Definition 4.5] *A quantum seed is a pair (M, B) where*

- *M is a toric frame in $\mathfrak{F}$.*
- *B is a $m \times n$-integer matrix with rows labeled by $[1, m]$ and columns labeled by an n-element subset* $\mathbf{ex} \subset [1, m]$.
- *The pair (B, Λ_M) is compatible.*

Now we define the seed mutation in direction of an exchangeable index $k \in \mathbf{ex}$. For each $\varepsilon \in \{1, -1\}$ we define a mapping $M_k : \mathbb{Z}^m \to \mathfrak{F}$ via

$$M_k(c) = \sum_{p=0}^{c_k} \binom{c_k}{p}_{q^{d_k} 2} M(E_\varepsilon c + \varepsilon p b^k) \ , \quad M_k(-c) = M_k(c)^{-1} \ ,$$

where we use the well-known q-binomial coefficients(see e.g. [5, Equation 4.11]), and the matrix $E_{k,\varepsilon}$ defined in (2.4). Define B_k to be obtained from B by the standard matrix mutation in direction k, as in Section 2.1. One obtains the following fact.

Proposition 4.3. [5, Prop. 4.7] *(a) The map M_k is a toric frame, independent of the choice of sign ε.*
(b) The pair (B_k, Λ_{M_k}) is a quantum seed.

Now, given an *initial quantum seed* (B, Λ_M) denote, in a slight abuse of notation, by $X_1 = M(e_1), \ldots, X_r = M(e_r)$, which we refer to as the *cluster variables* associated to the quantum seed (M, B). Here our nomenclature differs slightly from

[5], since there one considers the coefficients not to be cluster variables. This is, however, not useful for our purposes. We now define the seed mutation

$$X'_k = M(-e_k + \sum_{b_{ik}>0} b_{ik}e_i) + M(-e_k - \sum_{b_{ik}<0} b_{ik}e_i) \ .$$

We obtain that $X'_k = M_k(e_k)$ (see [5, Prop. 4.9]). We say that two quantum seeds (M, B) and (M', B') are mutation-equivalent if they can be obtained from one another by a sequence of mutations. Since mutations are involutive (see [5, Prop 4.10]), the quantum seeds in $\mathfrak{F}$ can be grouped in equivalence classes, defined by the relation of mutation equivalence. The quantum cluster algebra generated by a seed $(M, B) \subset \mathfrak{F}$ is the $\mathbb{C}[t^{\pm 1}]$-subalgebra generated by the cluster variables associated to the seeds in an equivalence class.

Note that there exists a special class of cluster variables. The cluster variables X_i with $i \in [1, m] - \mathbf{ex}$ never change under mutation. These cluster variables are called the *coefficients* of the quantum cluster algebra.

4.2. **The classical limit of a quantum cluster algebra.** In this section let $A = \mathbb{C}[t^{\pm 1}]$. Lt $\mathfrak{A}_t$ be the quantum cluster algebra defined by the quantum seed $(\mathbf{x}, B, \Lambda)$.

Consider the cluster algebra $\mathfrak{A}$ defined by the seed $(x_1, \ldots, x_n, B) \subset \mathbb{C}(x_1, \ldots, x_n)$. We have the following fact.

Proposition 4.4. *The classical limit of $\mathfrak{A}_t$ is the cluster algebra $\mathfrak{A}$.*

Proof. We have to find a basis for the A-lattice such that none of the monomials in the cluster variables (NOT cluster monomials in the sense of [16]) are contained in the maximal ideal $\mathfrak{m}_1$ generated by $t - 1$. However, Proposition 6.8 of [5] shows that forgetting about the quantum structure identifies the cluster variables of the quantum and the classical cluster algebra. Thus choose any $\mathbb{C}$-basis of the cluster algebra $\mathfrak{A}$. It defines a basis of $\mathfrak{A}_t$. Any monomial of quantum cluster variables in $\mathfrak{A}_t$ can be expressed as a $\mathbb{C}$-linear combination in the basis elements. Notice that the filtration induced by the powers of the maximal ideals $\mathfrak{m}_q$ generated by $(t - q)$ for $q \in \mathbb{C}^*$ is Hausdorff, i.e. $\cap_{n=0}^{\infty} \mathfrak{m}_q \mathfrak{A}_t = \{0\}$. We obtain now that the quantum cluster variables lie in $\mathfrak{A}_t - \mathfrak{m}_q \mathfrak{A}_t$. Thus, choosing $q = 1$,the cluster algebra $\mathfrak{A}$ is the classical limit of $\mathfrak{A}_q$. $\square$

Remark 4.5. *In particular, we can speak of a cluster $\overline{\mathbf{x}} \subset \mathfrak{A}$ as the classical limit $\overline{\mathbf{x}}$ of a quantum cluster $\mathbf{x} \subset \mathfrak{A}_q$.*

As in the classical case (see Section 2.5), we have notions of local and global toric actions for quantum cluster algebras. Suppose that $(\mathbf{x})$ is a quantum cluster in $\mathfrak{A}_q$ and its classical limit $(\overline{\mathbf{x}})$ a cluster in $\mathfrak{A}_q$. We define for each element $w = (w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n$ a *local toric action* of $\mathbb{C}^*$ via maps : $(x_1, \ldots, x_n) \mapsto (\alpha^{w_1} x_1, \ldots \alpha^{w_n} x_n)$ for all $\alpha \in k^*$. Assume now that we have chosen integer weights $w_{\mathbf{x}} = (w_1, w_2, \ldots, w_n)$ for each cluster $\mathbf{x}$. We say that the local toric actions are compatible if the following diagram commutes for any two clusters

$\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_i, \ldots, y_n)$

$$\begin{tikzcd} k[\mathbf{x}] \arrow[r, "T"] \arrow[d, "\psi_{\mathbf{x},\alpha}"] & k[\mathbf{y}] \arrow[d, "\psi_{\mathbf{y},\alpha}"] \\ k[\mathbf{x}] \arrow[r, "T"] & k[\mathbf{y}] \end{tikzcd} \quad ,$$

where T is a series of mutations transforming $\mathbf{x}$ to $\mathbf{y}$. Compatible local toric actions define a global toric action on the cluster algebra and a toric flow on the cluster variety. We have the following obvious consequence of Proposition 4.4 and [5, Proposition 6.8] .

Lemma 4.6. *Local and global actions commute with the classical limit.*

This implies that there is an action of an algebraic torus $H = (\mathbb{C}^*)^k$ simultaneously on $\mathfrak{A}_q$ and $\mathfrak{A}$.

4.3. **Our Example: The quantum Grassmannian $\mathbb{C}_q[G(2,5)]$.** Analogous to the classical case, we introduce the quantum Grassmannian $\mathbb{C}_q[2,5]$ and endow it with a quantum cluster algebra structure, following Grabowski [28]. Recall the definition of the algebra of quantum matrices $\mathbb{C}_q[Mat_{2,n}]$ (see also Section 5.2). Additionally recall the definition of the quantum 2×2-minors for $1 \le i < j \le n$ as

$$\Delta^q_{ij} = x_{1i}x_{2j} - qx_{1j}x_{2i} \ .$$

The quantum Grassmannian $\mathbb{C}_q[G(2,5)]$ is the subalgebra of $\mathbb{C}_q[Mat_{2,n}]$ generated by the quantum 2×2-minors which are subject to the quantum Plücker relations:

$$\Delta^q_{ik}\Delta^q_{j\ell} = \Delta^q_{ij}\Delta^q_{k\ell} + \Delta^q_{i\ell}\Delta^q_{jk} \ , \tag{4.1}$$

for $1 \le i < j < k < \ell \le 5$. The algebra $\mathbb{C}_q[G(2,5)]$ admits a quantum cluster algebra structure with initial cluster

$$\boldsymbol{\Delta_{13}}^q,\ \boldsymbol{\Delta_{14}}^q, \quad \Delta^q_{12},\ \Delta^q_{23},\ \Delta^q_{34},\ \Delta^q_{45},\ \Delta_{15} \ .$$

Again, as in the classical case, the last five variables are the coefficients. The compatible pair (B, Λ) of the quantum seed consists of the matrix B defined in (2.3) and Λ defined in (2.6). The quantum minors form the set of cluster variables, just as in the case of $\mathbb{C}[G(2,5)]$. A straightforward computation shows that $\mathbb{C}[G(2,5)]$ is the classical limit of $\mathbb{C}_q[G(2,5)]$.

4.4. **The Spectrum of a Quantum Cluster Algebra.** In this section we choose t to be a transcendental complex number q, i.e. we consider the quotient of $\mathfrak{A}_t$ by the $\mathbb{C}[t^{\pm 1}]$-module $(t-q)\mathfrak{A}_t$. We write $\mathfrak{A}_t = \mathfrak{A}_q/(t-q)\mathfrak{A}_t$ and refer to it as the quantum cluster algebra $\mathfrak{A}_q$. Notice that, as remarked in the proof of Proposition 4.4, none of the cluster variables are contained in $(t-q)\mathfrak{A}_t$. Additionally, notice that the quotient is compatible with the toric actions. We can compare the prime spectrum of a quantum cluster algebra $\mathfrak{A}_q$ to the Poisson prime spectrum of a cluster algebra $\mathfrak{A}$.

Main Theorem 4.7. *Let $\mathfrak{A}_q$ be a quantum cluster algebra, $\mathfrak{A}$ its classical limit and H the torus of global toric actions. The classical limit induces a H-equivariant homeomorphism $spec(\mathfrak{A}_q) \to P.spec(\mathfrak{A})$.*

Proof.

Denote by $\mathfrak{A}_q^\ell(\mathbf{x})$ the subalgebra of $\mathfrak{A}_q$ which is generated by all the cluster variables which are obtained from $\mathbf{x}$ by up to ℓ mutations. We shall, when there is no confusion about the initial cluster, write $\mathfrak{A}_q^\ell$ instead of $\mathfrak{A}_q^\ell((\mathbf{x})$ Recall the definition of the algebras $\mathfrak{A}^\ell$ for the classical limit $\mathfrak{A}$ from Section 3.3. The following facts are obvious corollaries from Proposition 4.4.

Lemma 4.8. *(a) The classical limit of $\mathfrak{A}_q^\ell$ is $\mathfrak{A}^\ell$.*
(b) The torus H acts on both $\mathfrak{A}_q^\ell$ and $\mathfrak{A}^\ell$, and the toric actions commute with the classical limit.

We prove the following Noetherian analogue of Theorem 4.7.

Theorem 4.9. *(a) Let $\ell \geq n$, the number of exchangeable indices. There exists an H-equivariant homeomorphism from*

$$spec(\mathfrak{A}_q^\ell) \cong P.spec(\mathfrak{A}^\ell) \ .$$

(b) Moreover, $spec(\mathfrak{A}_q^\ell) \cong spec(\mathfrak{A}_q^{\ell'})$ and $P.spec(\mathfrak{A}^\ell) \cong P.spec(\mathfrak{A}^{\ell'})$ for all $\ell, \ell' \geq n$.

Proof. Recall that by Goodearl-Letzter Stratification theory (Theorem B.3 and Theorem B.4) we obtain that $spec(\mathfrak{A}_q^\ell)$ is stratified as

$$spec(R) = \bigsqcup_{\mathcal{I} \in H-spec(\mathfrak{A}_q^\ell)} spec_J(\mathfrak{A}_q^\ell)$$

where $\mathcal{I} \in spec_J(\mathfrak{A}_q^\ell)$ implies that $(\mathcal{I} : H) = J$, and that $spec_J(\mathfrak{A}_q^\ell)$ is homeomorphic to the spectrum of a Laurent polynomial ring.

Thus, we first have to describe the torus invariant prime ideals $spec_H(\mathfrak{A}_q^\ell)$ in $\mathfrak{A}_q^\ell$. We have the following fact.

Proposition 4.10. *The classical limit induces a homeomorphism*

$$spec_H(\mathfrak{A}_q^\ell) \cong P.spec_H(\mathfrak{A}^\ell) \ .$$

Proof. Define the set $Y = \{x_1, \ldots, x_, n y_1, \ldots, y_m\}$ analogous to the classical case, by choosing y_i to be the cluster variable obtained from x_i by mutation at i.

We define a map $spec_H(\mathfrak{A}_q^\ell) \to P.spec_H(\mathfrak{A}^\ell)$ by sending a torus invariant prime ideal $\mathcal{I} \subset \mathfrak{A}_q^\ell$ to the toric prime ideal $\mathcal{I}_S \subset \mathfrak{A}^\ell$ containing $S = \mathcal{I} \cap Y$. We have to verifiy that this map is well defined,i.e. that $S \in PP_Y$. Suppose not. Then S violates the prime ideal condition of Proposition 3.2, and there exists an index i such that $x_i y_i \in \mathcal{I}$ but neither $x_i \in \mathcal{I}$ nor $y_i \in \mathcal{I}$. Without loss of generality, assume that $i = 1$. We have $\mathfrak{A}_q^\ell \subset R$ where R is the $\mathbb{C}$ algebra generated by $x_1, y_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}$ subject to the obvious relations (see [5]). Clearly, $x_1 r y_1 = \sum_\alpha s_\alpha x_1 y_1 t_\alpha \in \mathcal{I}$ for all $r \in \mathfrak{A}_q^\ell$ and some $s_\alpha, t_\alpha \in \mathfrak{A}_q^\ell$. This implies that either $x_1 \in \mathcal{I}$ or $y_1 \in \mathcal{I}$, a contradiction.

We now show that it is surjective. Suppose not. Let $S \subset Y$ define a classical toric Poisson prime ideal $\overline{\mathcal{I}}_S$ but not a quantum one. Apply mutations to obtain a cluster $\mathbf{x}'$, which is defining for the classical ideal $\overline{\mathcal{I}}_S$. Consider the set Y', defined with respect to $\mathbf{x}'$ and suppose that $\overline{\mathcal{I}}_S \cap Y' = S'$. If $\mathcal{I}$ is an ideal in $\mathfrak{A}_q^\ell$ such that $\mathcal{I} \cap Y' \supsetneq S'$, then $\mathcal{I} \cap \mathbf{x}' \subsetneq S \cap \mathbf{x}'$. Now suppose that $\mathcal{I}$ is a minimal prime containing the ideal $\mathcal{J}_S$ generated by S. It must contain some $x'_j \in \mathbf{x}'$ with $x'_j \notin S'$, by the definition of defining clusters. The ideal $\mathcal{I}$ is prime, hence if $arb \in \mathcal{I}$ for

all $r \in \mathfrak{A}_q^\ell$, then $a \in \mathcal{I}$ or $b \in \mathcal{I}$. Hence if every minimal prime contains some $x'_j \in \mathbf{x}'$ with $x'_j \notin S'$, then the ideal $\mathcal{J}_S$ generated by S must contain a polynomial $f \in \mathbb{C}_{\Lambda_\mathbf{i}}[x_j : x_j \notin S]$. That, however, is a contradiction, since the x_i, $i = 1, \ldots, n$ are algebraically independent.

Now, we prove that the map is injective. Suppose not. Then there exists $S \subset Y$ such that there are two toric prime ideals $\mathcal{I}^1 \neq \mathcal{I}^2$ which are minimal primes over $\mathcal{J}_S$. This implies that there exists a relation $f = ab \in \mathcal{J}_S$ with $a \in \mathcal{I}^1$ and $b \in \mathcal{I}^2$. Consider the ideal $\hat{\mathcal{J}}_S$ generated by S in $\mathfrak{A}_t^\ell$ (over $\mathbb{C}[t^{\pm 1}]$) Since there exists only one toric Poisson prime over S in the classical limit $\mathfrak{A}^\ell$, but $\mathcal{I}^1 \neq \mathcal{I}^2$, we obtain that there exists a pre-image $\hat{f} = \hat{a}\hat{b} \in \mathfrak{A}_t^\ell$ of $f = ab \in \mathfrak{A}_q^\ell$ such that $\hat{f} \in (t-1)^k \mathfrak{A}_t^\ell$ for some $k \geq 1$ but $\frac{\hat{f}}{t-1} \notin \hat{\mathcal{J}}_S$. Notice again that the filtration induced by the ideal generated by $(t-1)$ is Hausdorff and that $\hat{\mathcal{J}}_S$ is generated by elements in $\mathfrak{A}_t^\ell - (t-1)\mathfrak{A}_t$. We observe that if $\hat{f} \in \hat{\mathcal{J}}_S$ and $\hat{f} \in (t-1)^k \mathfrak{A}_t^\ell$ for some $k \geq 0$, then $\frac{\hat{f}}{t-1} \in \hat{\mathcal{J}}_S$. We obtain the desired contradiction. Injectivitiy is proved and it follows that the maps is a bijection.

In order to prove that the map is a homeomorphism with respect to the Zariski-type topology we need the following fact.

Lemma 4.11. [21, Lemma 9.4] *Let A be a noetherian $\mathbb{C}$-algebra and R a commutative noetherian Poisson k-algebra and let H be a complex torus acting by automorphisms on both algebras. A bijection $\Phi : spec_H(A) \to P.spec_H(R)$ is a homeomorphism if and only if Φ and Φ^{-1} preserve inclusions.*

Proof. The lemma is a H-equivariant version of [21, Lemma 9.4(a)], and is proved analogously. □

It is easy to observe, in both the quantum and the classical case, that $\mathcal{I}'_S \subset \mathcal{I}_S$, respectively $\overline{\mathcal{I}}'_S \subset \overline{\mathcal{I}}_S$ if and only if $S' \subset S$. Hence the proposition follows from Lemma 4.11. □

Remark 4.12. *This is the only place where we use the classical limit in the construction of the spectra. In the case of most applications to quantized coordinate rings, the bijections, and homeomorphsims are already known by other means, e.g. Yakimov's results in* [45] *and* [49].

Now consider a torus invariant prime ideal, defined by $S \in PP_Y$. Employing Algorithm 3.7, we may assume that $\mathbf{x}$ is defining for S. Analogous to the Poisson torus $H_{\Lambda_\mathbf{i}}$ we define the quantum torus $H^q_{\Lambda_\mathbf{i}}$ as the quantum torus generated by the cluster variables $x_j \notin S$, $1 \leq j \leq n$. Additionally, we use the notation $\mathbb{C}_\Lambda[M]$ for the non-commutative algebra generated by the elements of a set M of cluster variables of $\mathfrak{A}$, when the commutation relations are defined by Λ. For example $\mathbb{C}_\Lambda[x_1, \ldots, x_n] = \mathfrak{A}_q^0$ and $\mathbb{C}_\Lambda[x_1, \ldots, x_n, y_1, \ldots, y_m] = \mathfrak{A}_q^1$. Finally if J_S is a torus invariant prime ideal then we use the notation $spec_S \mathfrak{A}_q^\ell$for $spec_{J_S} \mathfrak{A}_q^\ell$. We have the following fact.

Theorem 4.13. *Let $S \in PP_Y$ and let $\mathcal{I}_S$ be the corresponding torus invariant prime ideal, and assume that $\mathbf{x}$ is a defining cluster with $\mathbf{i} = S \cap \mathbf{x}$. Then*

$$spec_S \mathfrak{A}_q^\ell = spec(H_\mathbf{i}^q)\ .$$

Proof. Recall from Goodearl-Letzter stratification theory (Theorem B.4) that

$$spec_S \mathfrak{A}_q^\ell \cong spec(\mathfrak{A}_q^\ell/\mathcal{I}_S[\mathcal{E_S}^{-1}])\ ,$$

where $\mathcal{E_S}^{-1}$ denotes the localization at the set of eigenvectors for the torus action in $\mathfrak{A}_q^\ell/J_Y$.

Recall that the global toric action on $\mathfrak{A}_q$ acts on $H^q_{\Lambda_\mathbf{i}}$. We need the following fact.

Proposition 4.14. *There exist natural embeddings*

$$H^q_{\Lambda_\mathbf{i}} \hookrightarrow \mathfrak{A}_q^\ell/\mathcal{I}_S[\mathcal{E_S}^{-1}] \hookrightarrow H^q_{\Lambda_\mathbf{i}}[\mathcal{E}_{*\mathbf{i}}^{-1}]\ ,$$

where $\mathcal{E}_{*\mathbf{i}}$ denotes the set of eigenvectors for the global toric actions on $H^q_{\Lambda_\mathbf{i}}$.

Proof. The first inclusion is obtained from the embedding of $\mathbb{C}_{\Lambda_\mathbf{i}}[x_j : j \notin \mathbf{i}]$ into $\mathfrak{A}_q^\ell/\mathcal{I}_S$. The second inlcusion follows from th e following fact.

Lemma 4.15. *Let Y, $\mathbf{x}$, $\mathbf{i}$ and $\mathcal{I}_S$ be as above. Then each element $z \in \mathfrak{A}_q^\ell/\mathcal{I}_S$ has a representative $\tilde{z} \in \mathfrak{A}_q^\ell$ such that $\tilde{z} \in H^q_{\Lambda_\mathbf{i}}$.*

Proof. Recall that by definition, for each exchangeable index $i \in [1,n]$ we have $\mathfrak{A}_q^\ell \subset \mathbb{C}[x_1^{\pm 1} \ldots, x_{i-1}^{\pm 1}, x_i, y_i, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$. Now, suppose that $z \in \mathfrak{A}_q^\ell/J_Y$ has no representative in $\tilde{z} \in H^q_{\Lambda_\mathbf{i}}$. Choose any representative z' and $x_i \in S$ and express z' as a polynomial in $\mathbb{C}[x_1^{\pm 1} \ldots, x_{i-1}^{\pm 1}, x_i, y_i, x_{i+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Recall the following fact which is adapted from [50]. Suppose that $u \in \mathfrak{A}_q^\ell$ is a Laurent polynomial in $\mathbb{C}_\Lambda[x_{k_1}^{\pm 1}, x_{k_r}^{\pm 1}]$ for some set $\{k_1, \ldots, k_r\} \subset [1,n]$. We want to show that $u \in \mathbb{C}_\Lambda[x_{k_1}, y_{k_1}, x_{k_2}^{\pm 1}, \ldots, x_{k_r}^{\pm 1}]$. Suppose not. We have $u = f + y_{k_1} g$ where $f \in \mathbb{C}_\Lambda[x_{k_1}, x_{k_2}^{\pm 1}, \ldots, x_{k_r}^{\pm 1}]$ and $g \in \mathbb{C}_\Lambda[y_{k_1}, x_{k_2}^{\pm 1}, \ldots, x_{k_r}^{\pm 1}]$ with $g \neq 0$. Our assumption implies that $x_{k_1} y_{k_1} = x_w h + h'$ for some $w \notin \{k_1, \ldots, k_r\}$ and $h, h' \in \mathbb{C}_\Lambda[\mathbf{x}]$. If, however, we express u as a Laurent polynomial in $H^q_{\Lambda_\mathbf{i}}$, we obtain a term involving $x_{k_1}^{-1} x_w$ and the desired contradiction.

Inductively, we can now write $z' = x_{i_1} f_1 + y_{i_1} g_1 + \ldots + x_{i_r} f_r + y_{i_r} g_r + \tilde{z}$ where $f_{i_k} \in \mathbb{C}_\Lambda[x_j^{\pm 1}, x_{i_k} : j \notin \{i_1, \ldots, i_k\}]$, $g_{i_k} \in \mathbb{C}_\Lambda[x_j^{\pm 1}, y_{i_k} : j \notin \{i_1, \ldots, i_k\}]$ and $\tilde{z} \in H^q_{\Lambda_\mathbf{i}}$. The element $\tilde{z}$ is the desired representative. The lemma is proved. □

The proposition is proved. □

We obtain that if z is central in $H^q_{\omega_i}$ then it is central in $\mathfrak{A}_q^\ell/J_S[\mathcal{E_S}^{-1}]$, and if z' is central in $\mathfrak{A}_q^\ell/J_S[\mathcal{E_S}^{-1}]$ then it is also central in $H^q_{\Lambda_\mathbf{i}}[\mathcal{E}_{*\mathbf{i}}^{-1}]$. Therefore we have embeddings of centers which induce, by Theorem B.4 embeddings of the spectra

$$Z(H^q_{\omega_i}) \overset{f}{\hookrightarrow} Z(\mathfrak{A}_q^\ell/J_S[\mathcal{E_S}^{-1}]) \overset{g}{\hookrightarrow} Z(H^q_{\Lambda_\mathbf{i}}[\mathcal{E}_{*\mathbf{i}}^{-1}])\ .$$

$$spec(H^q_{\omega_i}) \overset{\tilde{f}}{\hookrightarrow} spec(\mathfrak{A}_q^\ell/J_S[\mathcal{E_S}^{-1}]) \overset{\tilde{g}}{\hookrightarrow} spec(H^q_{\Lambda_\mathbf{i}}[\mathcal{E}_{*\mathbf{i}}^{-1}])\ .$$

We need the following fact.

Lemma 4.16. *The map $gf : Z(H^q_{\omega_i}) \to Z(H^q_{\Lambda_\mathbf{i}}[\mathcal{E}_{*\mathbf{i}}^{-1}])$ satisfies $gf(Z(H^q_{\omega_i})) = Z(H^q_{\Lambda_\mathbf{i}}[\mathcal{E}_{*\mathbf{i}}^{-1}])$.*

Proof. The generators of both $Z(H^q_{\omega_i})$ and $Z(H^q_{\Lambda_\mathbf{i}}[\mathcal{E}_{*\mathbf{i}}^{-1}])$ are of linearly independent weights by Theorem B.4 (e). But we know from Theorem B.4 that $\tilde{g}\tilde{f}$ is an

isomorphism of torus graded Laurent polynomial rings. Hence the embedding must be surjective. The lemma is proved. □

We immedtely obtain the desired homeomorphism or rather equality of the centers. Theorem 4.13 is proved. □

4.4.1. *Homeomorphisms.* We have thus established the set theoretical bijections. We now have to prove that they are actually homeomorphisms. Again we apply Lemma 4.11 with trivial torus, and show that the classical limit induces inclusion preserving bijections on the spectra.

Now let $(\mathbf{x}, B, \Lambda)$ be a quantum seed and let $S_1, \dots S_r \in PP_Y$ denote the elements of PP_Y for which $\mathbf{x}$ is a defining cluster. As above, let $\mathbf{i}_j = S_j \cap \mathbf{x}$. There are natural embeddings (see also Lemma A.3)

$$\bigsqcup_{j=1}^{r} spec_{S_j}(\mathfrak{A}_q^{\ell}) \cong \bigsqcup_{j=1}^{r} spec(H_{\mathbf{i}_j}^{q}) \hookrightarrow spec(\mathbb{C}_\Lambda[\mathbf{x}]) \ ,$$

$$\bigsqcup_{j=1}^{r} P.spec_{S_j}(\mathfrak{A}^{\ell}) \cong \bigsqcup_{j=1}^{r} P.spec(H_{\mathbf{i}_j}) \hookrightarrow P.spec(\mathbb{C}[\mathbf{x}], \Lambda) \ .$$

We have the following fact.

Lemma 4.17. *The classical limit induces a homeomorphism*

$$\bigsqcup_{j=1}^{r} spec_{S_j}(\mathfrak{A}_q^{\ell}) \cong \bigsqcup_{j=1}^{r} P.spec_{S_j}(\mathfrak{A}^{\ell}) \ .$$

Proof. Consider $\mathbb{C}_\Lambda[\mathbf{x}] \subset \mathfrak{A}_q^{\ell}$ and $(\mathbb{C}[\mathbf{x}], \Lambda \subset \mathfrak{A}^{\ell}$. The bijection, defined by the classical limit $spec(\mathbb{C}_\Lambda[\mathbf{x}]) \to P.spec(\mathbb{C}[\mathbf{x}], \Lambda)$ (see Proposition A.2(b)) induces a bijection $\bigsqcup_{j=1}^{r} spec(H_{\mathbf{i}_j}^{q}) \to \bigsqcup_{j=1}^{r} P.spec(H_{\mathbf{i}_j})$. Now let $\mathcal{I}, \mathcal{I}' \in \bigsqcup_{j=1}^{r} spec_{S_j}(\mathfrak{A}_q^{\ell})$. Observe that $\mathcal{I} \subset \mathcal{I}'$ if and only if $(\mathcal{I} \cap \mathbb{C}_\Lambda[\mathbf{x}]) \subset (\mathcal{I}' \cap \mathbb{C}_\Lambda[\mathbf{x}])$ (see also Appendix A and Lemma A.3). Analogously, let $\overline{\mathcal{I}}, \overline{\mathcal{I}}' \in \bigsqcup_{j=1}^{r} P.spec_{S_j}(\mathfrak{A}^{\ell})$. Again, we obtain that $\overline{\mathcal{I}} \subset \overline{\mathcal{I}}'$ if and only if $\left(\overline{\mathcal{I}} \cap \mathbb{C}[\mathbf{x}]\right) \subset \left(\overline{\mathcal{I}}' \cap \mathbb{C}[\mathbf{x}]\right)$. Hence the classical limit and its inverse preserve inclusions. Applying Lemma 4.11 yields the assertion. □

Recall from the proof of Proposition 4.10 that if $\mathcal{I} \subset \mathcal{I}'$ are two ideals, $\mathcal{I} \in spec_{J_S}(\mathfrak{A}_q^{\ell})$ and $\mathcal{I}' \in spec_{J_{S'}}(\mathfrak{A}_q^{\ell})$, then $S \subset S'$. Additionally, we have an analogous statement for the elements of $P.spec(\mathfrak{A}^{\ell})$.

If $S \subset S'$, then there exists a quantum cluster $\mathbf{x}$ which is simultaneously a defining cluster for J_S as well as for $J_{S'}$. Indeed, just apply Algorithm 3.7 to S' first and then to S. Hence, Lemma 4.17 yields Theorem 4.9(a).

For part(b) it suffices to observe that in order to analyze the spectrum of $\mathfrak{A}_q^{\ell}$ we only considered clusters that were obtained from $\mathbf{x}$ by fewer than n mutations. Hence the spectra of $\mathfrak{A}_q^{\ell}$ and $\mathfrak{A}_q^{\ell'}$ are homeomorphic for all $\ell, \ell' \geq n$. Part(b) follows and Theorem 4.9 is proved. □

4.4.2. *The proof of Theorem 4.7.* We now conclude the proof of Theorem 4.7, but we will first make the following remarks.

Remark 4.18. *Notice that if $\mathfrak{A}_q$ is Noetherian, then the assertion already follows from Theorem 4.9.*

Remark 4.19. *Let $\mathfrak{A}_q^\ell$ and $\mathfrak{A}_q^{\ell'}$ be as above. The homeomorphism of spectra is given by the map which sends an ideal $\mathcal{I} \subset \mathfrak{A}_q^\ell$ to $\mathcal{I}' = \mathcal{I} \cap \mathfrak{A}_q^{\ell'}$. Notice that, in general, such intersection maps do not preserve prime idelas in arbitrary non-commutative rings, however, our results show that it does in the case of quantum cluster algebras.*

Observe that $\mathfrak{A}_q = \bigcup_{\ell=1}^{\infty} \mathfrak{A}_q^\ell$ and $\mathfrak{A} = \bigcup_{\ell=1}^{\infty} \mathfrak{A}^\ell$. We have to show that if $\mathcal{I} \in spec(\mathfrak{A}_q)$ then $\mathcal{I}^\ell = \mathcal{I} \cap \mathfrak{A}_q^\ell$ is a prime ideal in $\mathfrak{A}_q^\ell$ for all $\ell \geq m$.

Suppose not. Then, Remark 4.19 implies that $\mathcal{I}^\ell$ is not a prime ideal for all $\ell \geq m$. This implies that there exist elements $a, b \in \mathfrak{A}_q^\ell$ such that $arb \in \mathcal{I}^\ell$ for all $r \in \mathfrak{A}_q^\ell$, but $asb \notin \mathcal{I}$ for some $s \in \mathfrak{A}$. However, since $s \in \mathfrak{A}_q^{\ell_r}$ for some $\ell_r \geq m$, we obtain the desired contradiction.

We claim that the map $\mathcal{I} \to \mathcal{I}^\ell$ induces a homeomorphism of spectra. Since each prime ideal $\mathcal{J}^\ell \in \mathfrak{A}_q^\ell$ is contained in a unique minimal prime ideal $\mathcal{J}^{\ell'}$ in $\mathfrak{A}_q^{\ell'}$ for each $\ell' \geq \ell$, we observe that $\mathcal{J} = \bigcup_{\ell' \geq \ell} \mathcal{J}^{\ell'}$ is a minimal prime ideal containing $\mathcal{J}^\ell$. Clearly, $\mathcal{J} \cap \mathfrak{A}_q^\ell = \mathcal{J}^\ell$, and therefore, the map is surjective. Now, suppose that the map is not injective. Then there exist prime ideals $\mathcal{I}^1, \mathcal{I}^2 \subset \mathfrak{A}_q$ such that $\mathcal{I}^1 \cap \mathfrak{A}_q^\ell = \mathcal{I}^2 \cap \mathfrak{A}_q^\ell$ for some $\ell \geq m$ and $\mathcal{I}^1 \neq \mathcal{I}^2$. But by our usual argumentation this implies that $\mathcal{I}^1 \cap \mathfrak{A}_q^{\ell'} = \mathcal{I}^2 \cap \mathfrak{A}_q^{\ell'}$ for all $\ell' \geq m$ and this implies that $\mathcal{I}^1 = \mathcal{I}^2$. Therefore, the intersection maps induce bijections of spectra. Since they clearly observe inclusion relations betrween the ideals, they are indeed homeomorphisms, as desired. Theorem 4.7 is proved. $\square$

Theorem 4.9 and Theorem 4.7 suggest the following conjecture, suggesting that the spectra of quantum cluster algebras are determined by their lower bounds.

Conjecture 4.20. *Let $\mathfrak{A}_q$ be a quantum cluster algebra and $(\mathbf{x}, B, \Lambda)$ a quantum seed. Then $spec(\mathfrak{A}_q)$ is homeomorphic to $spec(\mathfrak{A}_q^1)$, the lower bound.*

Recall from Theorem 3.12 that if $\mathfrak{A}$ is a finitely generated Poisson cluster algebra and $\mathfrak{A}$ is the algebra of functions on an affine complex variety X, then the symplectic leaves on X are algebraic, i.e. coincide with the Poisson cores. We obtain the following corollary of Theorem 4.7.

Corollary 4.21. *Let $\mathfrak{A}^q$ be a finitely generated quantum cluster algebra and $\mathfrak{A} = (\mathbb{C}[X], \{\cdot, \cdot\})$ its classical limit, where X is a suitable complex affine variety. There exists a homeomorphism between the set of primitive ideals in $\mathfrak{A}^q$ and the symplectic leaves in*

$$prim(\mathfrak{A}_q) \cong symp_{\{\cdot,\cdot\}}(X) \ .$$

Proof. The assertion follows from Theorem 4.7 by applying the following fact.

Lemma 4.22. [21, Lemma 9.4 (b), (c)] *Let A be a noetherian $\mathbb{C}$-algebra and R a commutative noetherian Poisson k-algebra.*
(a) Any homeomorphism $prim(A) \to P.prim(R)$ extends uniquely to a homeomorphism $spec(A) \to P.spec(R)$.
(b) On the other hand, any homeomorphism $spec(A) \to P.spec(r)$ restricts to a homeomorphism $prim(A) \to P.prim(R)$.

□

4.5. **Example.** In the case of the quantum Grassmannian $G(2,5)$, Launois, Lenagan and Rigal showed how to describe the torus invariant prime ideals using Cauchon diagrams [39]. However, explicitly computing the spectra by constructing defining clusters and analyzing the centers of the involved quantum tori, is a more involved problem which hopefully will be addressed in the future.

5. Applications

Recently, a number of functions on quantized coordinate rings over $\mathbb{C}[t^{\pm 1}]$ have been discovered to have quantum cluster algebra structures. We consider the specializations of the algebras involved at a transcendental number $t = q \in \mathbb{C}^*$ (see Appendix A).

5.1. **The algebras $\mathfrak{a}_t(\mathfrak{n}_w)$.** Let $\mathfrak{g}$ be a Kac-Moody Lie algebra over $\mathbb{C}$ and W its Weyl group, and $U_q(\mathfrak{g})$ its quantized enveloping algebra. Denote by $U_q(\mathfrak{n})$ its positive part. For each $w \in W$ we can define an algebra $\mathfrak{a}_q(\mathfrak{n}_w)$, which is isomorphic to $U_q(\mathfrak{n}_w)$, the quantum Schubert cell as an algebra (though making this isomorphism precise for the quantum cluster algebra structure is a slightly non-trivial issue). Following GeißLeclerc and Schröer [18], this algebra $\mathfrak{a}_q(\mathfrak{n}_w)$ can be viewed as a quantum deformation of the algebras $\mathbb{C}[N(w)]$, discussed in [16] and [17]. In the latter paper, they showed that $\mathbb{C}[N(w)]$ has a quantum cluster algebra structure . Recently, GeißLeclerc and Schröer proved in [18] that $\mathfrak{a}_q(\mathfrak{n})$ has a quantum cluster algebra structure. It is well-known that the Poisson structure induced by the classical limit on $\mathbb{C}[N(w)]$ is the standard Poisson-Lie structure, the torus is the maximal torus of G. Hence we obtain the following result from Corollary 4.21.

Theorem 5.1. *Let $q \in \mathbb{C}^*$ be transcendental. There exists a homeomorphism, induced by the classical limit, between $prim(\mathfrak{a}_q(\mathfrak{n}_w))$ and $symp(N(w))$.*

5.2. **Quantum matrices.** Let $q \in \mathbb{C}^*$ be transcendental. Recall the definition of the algebra of quantum matrices. The algebra $\mathbb{C}_q[M_{2\times 2}]$ of *quantum* 2×2-matrices is the $\mathbb{C}$-algebra generated by a, b, c, d subject to the relations

$$ca = qac,\ ba = qab,\ dc = qcd,\ db = qbd,\ cb = bc,\ da - ad = (q - q^{-1})bc\ .$$

We denote by $\mathbb{C}_q[M_{d\times k}]$ the algebra of *quantum* $m \times n$-matrices, i.e., the $\mathbb{C}$-algebra generated by x_{ij}, $1 \le i \le d$, $1 \le j \le k$ with the following defining relations: for each $1 \le i < i' \le d, 1 \le j < j' \le k$ the subalgebra of $\mathbb{C}_t[M_{m\times n}]$ generated by the four elements $a = x_{i,j}, b = x_{i,j'}, c = x_{i',j}, d = x_{i',j'}$ is isomorphic to $\mathbb{C}_q[M_{2\times 2}]$. Using the results on the cluster algebra structure on $U_q(\mathfrak{n}_w)$, Geiss, Leclerc and Schröer showed in [18] that $\mathbb{C}_q[M_{m\times n}]$ (as a $\mathbb{C}[q^{\pm 1}]$-algebra) has a quantum cluster algebra structure. Its classical limit is the well-known cluster algebra structure on $\mathbb{C}[Mat_{m\times n}]$ with the standard Poisson structure. It is well-known that the global toric actions are the standard toric actions of $(\mathbb{C}^*)^m \times (\mathbb{C}^*)^n$ on $\mathbb{C}[Mat_{m\times n}]$ via left multiplication by $m\times m$-diagonal matrices and right multiplication by $n\times n$-diagonal matrices (see the example in Section 2.6.2). We obtain the following result.

Theorem 5.2. *There exists a homeomorphism, induced by the classical limit, between the primitive ideals $prim(\mathbb{C}_q[Mat_{m\times n}])$ and the symplectic leaves $symp(Mat_{m\times n})$ with respect to the standard Poisson structure.*

5.3. **The primitive spectrum of $\mathcal{O}_q(GL_n)$.** Recall that the quantized coordinate ring $\mathcal{O}_q(GL_n)$ of GL_n can be defined as the localization of $\mathbb{C}_q[M_{n\times n}]$ at the normal element Δ_q where Δ_q is the quantum determinant

$$\Delta = \sum_{\sigma\in\mathfrak{S}_n} (-q)^{\ell(s)} x_{1\sigma(1)} \dots x_{n\sigma(n)} .$$

Similarly, the quantized coordinate ring $\mathcal{O}_q(SL_n)$ is obtained requiring $\Delta_q = 1$. The corresponding classical limits are the standard Poisson-Lie structures on GL_n, resp. SL_n. We have the following result.

Theorem 5.3. *There exist a torus equivariant homeomorphisms, induced by the classical limit, between*
(a) prim($\mathcal{O}_q(GL_n)$) and symp(GL_n).
(b) prim($\mathcal{O}_q(SL_n)$) and symp(SL_n).

Proof.
Notice that Δ_q is a coefficient in the quantum cluster algebra structure on $\mathbb{C}_q[Mat_{n\times n}]$. This follows, indirectly, from [18, Section 12], however, it also suffices to observe that Δ_q generates a torus invariant prime ideal (see e.g.[23] and [24]). Additionally, note the well-known fact that Δ_q is central. Now $\mathbb{C}_q[Mat_{n\times n}][\Delta_q^{-1}]$ has a quantum cluster algebra structure (with one invertible coefficient), as well as the quotient of $\mathbb{C}_q[Mat_{n\times n}]$ by the ideal generated $(\Delta_q - 1)$. In the latter case, the cluster algebra structure is obtained by simply removing the k-th column from the matrix B, and the k-th row and column from the matrix Λ. Hence, both $\mathcal{O}_q(GL_n)$ and $\mathcal{O}_q(SL_n)$ inherit a quantum cluster algebra structure. Corollary 4.21 yields the desired result. □

6. Open Questions

The most straightforward question regards categorical versions of our results. The main tool used to construct the quantum cluster algebras by GeißLeclerc and-Schröer [18] is to consider (additive) categorifications using certain categories of representations of preprojective algebras. To these representation varieties we can attach using the (motivic) Hall algebras, geometrical objects–sheaves on Grothendieck groups.

Question 6.1. *(a)What are the categorical interpretations of Poisson ideals and torus orbits of symplectic leaves?*
(b)What are the geometric interpretation, using categorifications?
(c) Describe the ideal theory of motivic Hall algebras attached to cluster algebras.

Recall that as noted in the introduction, the orbit method has been very successful in understanding the representation theory and geometry of Lie groups and algebras. In the classical instances, it is believed (see e.g. Kirillov [36]) that it works because of properties of the logarithm, in particular the Campbell-Baker-Hausdorff-formula. The quantizations of cluster algebras and cluster varieties are closely related to dilogarithm functions (see e.g. Fock and Goncharov's papers [10], [11] and Keller's preprint [34]). This leads to the following general question.

Problem 6.2. *Develop a dilogarithmic variant of the orbit method, including a representation theory of quantum cluster algebras.*

Appendix A. Families of Quantum Algebras and their Specializations

In this section we will recall some properties of the classical limit of a quantized function algebra. For further reference, we refer the reader to [6, Chapter III.5] and [21]. Let $A = \mathbb{C}[t^{\pm 1}]$, the ring of Laurent polynomials in one variable. It is well-known that its maximal ideals are generated by linear polynomials $(t-\alpha)$ where $\alpha \in \mathbb{C}^*$. Denote the maximal ideal generated by $(t-\alpha)$ by $\mathfrak{m}_\alpha$. For convenience we will use the notation $\mathfrak{m}_1 = \mathfrak{m}$.

Let $\mathfrak{A}$ be a finitely generated A-algebra. Let $q \in \mathbb{C}^*$. denote by $\mathfrak{A}_q = \mathfrak{A}/\mathfrak{m}_q\mathfrak{A}$, the quotient of $\mathfrak{A}$ by $(t-q)\mathfrak{A}$. This means, we choose $t = q$. If $q = 1$ we call $\mathfrak{A}_1$ the classical limit of $\mathfrak{A}$. Note the following fact.

Proposition A.1. *Suppose that $\mathfrak{A}/(\mathfrak{m}_1\mathfrak{A})$ is commutative.*
(a)Then, $\overline{\mathfrak{A}}$ is a Poisson algebra with Poisson bracket given for any $\bar{a}, \bar{b} \in \overline{\mathfrak{A}}$ and corresponding representatives $a, b \in \mathfrak{A}$ as

$$\{\bar{a}, \bar{b}\} = \frac{ab - ba}{q-1} mod(\mathfrak{m}_1\mathfrak{A}) \ .$$

(b) Moreover, let $\mathcal{I} \in spec_1(\mathfrak{A})$. Its image under the classical limit is a Poisson prime ideal.

For our argumentation it is of particular importance to note the spectral properties of the following classical limits. Let Λ be a skew-symmetric $n \times n$-integer matrix and H_Λ^q and H_Λ the corresponding quantum torus, respectively Poisson torus, i.e. the algebra

$$A(x_1^{\pm 1}, \dots, x_n^{\pm 1})/\mathcal{I} \ ,$$

where $\mathcal{I}$ is the ideal generated by the relation $x_i x_j = q^{\lambda_{ij}} x_j x_i$ for all $1 \leq i, j \leq n$, resp. the Poisson algebra $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with Poisson bracket defined by extending $\{x_i, x_j\} = \lambda_{ij} x_i x_j$.

Similarly define the quantum space $\mathbb{C}_\Lambda[x_1, \dots, x_n]$ as

$$A(x_1^{\pm 1}, \dots, x_n^{\pm 1})/\mathcal{I} \ ,$$

where $\mathcal{I}$ is the ideal generated by the relation $x_i x_j = q^{\lambda_{ij}} x_j x_i$ for all $1 \leq i, j \leq n$.

Denote by $P.spec(\mathfrak{A})$ the spectrum of Poisson prime ideals of a Poisson algebra $\overline{\mathfrak{A}}$ and by $P.prim(\overline{\mathfrak{A}})$ the spectrum of the Poisson primitive ideals, while $prim_1(\mathfrak{A})$ denotes the primitive 1-defined ideals of an A-algebra $\mathfrak{A}$. The following facts are well known and follow from various results in [21] and [22].

Proposition A.2. *(a) Let H_Λ^q and H_Λ be a quantum, resp. Poisson torus. Then, H_Λ is the classical limit of H_Λ^q. Moreover, $spec_1(H_\Lambda^q)$ is homeomorphic to $P.spec(H_\Lambda)$, and $prim_1(H_\Lambda^q)$ is homeomorphic to $P.prim(H_\Lambda)$.*

(b) The classical limit of $\mathbb{C}_\Lambda[x_1, \dots, x_n]$ is the Poisson algebra $(\mathbb{C}[x_1, \dots, x_n], \{\cdot,\cdot\})$ where $\{x_i, x_j\} = \lambda_{ij} x_i x_j$. Moreover, $spec_1(\mathbb{C}_\Lambda[x_1, \dots, x_n])$ is homeomorphic to $P.spec((\mathbb{C}[x_1, \dots, x_n], \{\cdot,\cdot\}))$, and analogously for the primitive and Poisson primitive spectra.

Finally, we apply Goodearl-Letzter stratifications to $\mathbb{C}_\Lambda[x_1, \dots, x_n]$ and obtain the following fact (using the notation of Section 3.3). Notice that the torus $(\mathbb{C}^*)^n$ acts naturally on $\mathbb{C}_\Lambda[x_1, \dots, x_n]$ via $\beta(x_i) = \beta_i x_i$ for all $\beta \in (\mathbb{C}^*)^n$ and $1 \leq i \leq n$.

Lemma A.3. *The torus invariant Poisson prime ideals J_S of $\mathbb{C}_\Lambda[x_1, \ldots, x_n]$ are the ideals generated by subsets of $S \subset \{x_1, \ldots x_n\}$. We have*

$$spec_1(\mathbb{C}_\Lambda[x_1, \ldots, x_n]) = \bigsqcup_{S \subset \{x_1, \ldots x_n\}} spec_{J_S,1}(\mathbb{C}_\Lambda[x_1, \ldots, x_n]) \cong \bigsqcup_{S \subset \{x_1, \ldots x_n\}} spec_1(H^q_{\Lambda_{S^c}}) \ ,$$

where S^c denotes the complement of S in $\{x_1, \ldots x_n\}$.

Remark A.4. *The classical limit of the quantum torus and the quantum affine plane are special with regards to their spectral properties, as the quantizations are constructed using cocycle twists (see e.g.*[6, I.1.16]

Appendix B. Torus Invariant prime ideals in non-commutative Noetherian rings

In this section we note some basic results from [6, Section II]. First, we recall the definition of a prime ideal in a non-commutative ring which differs slightly from the usual definition in a commutative ring. Let R be a ring. A two-sided ideal $\mathcal{I} \subset R$ is called a prime ideal if $arb \in \mathcal{I}$ for all $r \in R$, implies that $a \in \mathcal{I}$ or $b \in \mathcal{I}$. A two-sided ideal $\mathcal{I} \in R$ is called *primitive* if it is the maximal two-sided ideal contained in a left-maximal ideal. We denote the set of prime ideals by $spec(R)$ and the set of primitive ideals by $prim(R)$. Both sets come with a natural Zariski-type topology, where the closed sets are subsets of $spec(R)$, resp. $prim(R)$ such that $V(\mathcal{I}) = \{P \in spec(R) : P \supset \mathcal{I}\}$, resp. $V(\mathcal{I}) = \{P \in prim(R) : P \supset \mathcal{I}\}$.

Now let H be a group acting by automorphisms on R. H permutes the prime, primitive and left-maximal ideals of R. An ideal is called H-stable if $H(\mathcal{I}) = \mathcal{I}$ and, following the notation of [6, Section II.1] we denote by $(\mathcal{I} : H)$ the largest H-stable ideal contained in H, i.e.

$$(\mathcal{I} : H) = \bigcap_{h \in H} h(\mathcal{I}) \ .$$

We can now define the key notion of an H-prime ideal. The ring R is called H-prime, if R is nonzero and any product of nonzero H-ideals is nonzero.

Definition B.1. *A proper H-ideal $\mathcal{I}$ is called an H-prime ideal if $R/\mathcal{I}$ is an H-prime ring.*

If $\mathcal{I}$ is a prime ideal, then it is easy to see that $(\mathcal{I} : H)$ is H-prime. Note that in general a H-prime ideal need not be prime (see [6, section II.1.9]), however in the case we are most interested in, we have the following fact.

Lemma B.2. [6, Corollary II.1.12] *Let R be a Noetherian ring and let k be an algebraically closed field, and suppose that H is a k-torus acting on R by automorphisms (R is not necessarily a k-algebra). Then all H-prime ideals of R are prime.*

Denote the set of H-prime ideals of R by $H - spec(R)$. We naturally obtain a stratification of $spec(R)$ by the H-prime ideals

$$spec(R) = \bigsqcup_{J \in H - spec(R)} spec_J(R) \ ,$$

where $spec_J(R) = \{P \in spec(R) : (P : H) = J\}$. Goodearl and Letzter established in [25] a general description of the prime and primitive spectra of a large

class of Noetherian algebras which encompass many quantized coordinate rings. In particular, they obtain the following results.

Theorem B.3. *(a)* [6, Theorem II.8.10] *Let A be a Noetherian algebra over the complex numbers, and let H be a complex affine algebraic group, acting rationally on A by algebra automorphisms. Assume that $H - spec(A)$ is finite. Then, the primitive ideals are exactly the ideals which are maximal in their H-strata.*

(b) [6, Theorem II.8.14] *Moreover, the H-orbits within $prim(A)$ coincide with the H-strata of $prim(A)$. In particular there are only finitely many H-orbits in $prim(A)$.*

Moreover, we have the following fact which allows us to classify the primitive ideals in each stratum.

Theorem B.4. [6, Theorem II.2.13] *Let A be a Noetherian k-algebra, k infinite, and let H be an algebraic torus acting rationally on A by k-algebra automorphisms, and let $J \in H - spec(A)$.*

(1) *J is a prime ideal.*
(2) *Let $\mathfrak{E}_J$ denote the set of all regular H-eigenvectors in A/J. Then $\mathfrak{E}_J$ is a denominator set, and the localization $A_J = (A/J)[\mathfrak{E}_J^{-1}]$ is an H-simple ring with respect to the induced H action.*
(3) *$spec_J(A)$ is homeomorphic to $spec(A_J)$ via localization and contraction.*
(4) *$spec(A_J)$ is homeomorphic to $spec(Z(A_J))$ via contraction and extension, where $Z(A_J)$ denotes the center of A_J.*
(5) *$Z(A_J)$ is a Laurent polynomial ring. The indeterminates can be chosen to be H-eigenvectors with linearly independent H-eigenvalues.*

proved.

References

[1] J. Bell, K. Casteels and S. Launois, Primitive Ideals in Quantum Schubert cells:dimension of the strata, preprint arXiv:1009.1347.
[2] J. Bell and S. Launois, On the Dimension of H-strata in Quantum Matrices, Algebra and Number Theory 4 (2010) 175–200.
[3] J. Bell, S. Launois and N.Nguyen, Dimension and Enumeration of Primitive Ideals in Quantum Algebras, J. Algebr. Comb. 29 (2009), 269–294.
[4] A. Berenstein, S. Fomin and A. Zelevinsky, Cluster Algebras III: Upper Bounds and Double Bruhat Cells. Duke Math . J. 126 (2005), no. 1, 1–52.
[5] A. Berenstein and A. Zelevinsky, Quantum Cluster Algebras, Adv. Math. 195 (2005), no. 2, 405–455.
[6] K. Brown and K. Goodearl, Lectures on Algebraic Quantum Groups, Birkhäuser, 2002.
[7] K. Brown, K. Goodearl and M. Yakimov, Poisson Structures on Affine Spaces and Flag Varieties. I. Matrix Affine Poisson Space, Adv. in Math. 206 (2006) 567–629.
[8] K. Brown and I. Gordon, Poisson orders, symplectic reflection algebras and representation theory. J. Reine Angew. Math. 559 (2003), 193–216.
[9] G. Cauchon and A. Meriaux, Admissible Diagrams in $U_q^w(\mathfrak{g})$ and Combinatoric Properties of Weyl Groups, preprint arXiv:0902.0754.
[10] V. Fock, A. Goncharov, The quantum dilogarithm and representations of quantum cluster varieties. Invent. Math. 175 (2009), no. 2, 223?286.
[11] V. Fock, A. Goncharov,Cluster ensembles, quantization and the dilogarithm. Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 6, 865?930.
[12] S. Fomin, M. Shapiro and D. Thurston, Cluster algebras and Triangulated Surfaces. I. Cluster complexes, Acta Math. 201 (2008), no. 1, 83–146.
[13] S. Fomin and D. Thurston, Cluster algebras and Triangulated Surfaces. Part II: Lambda Lengths, http://www.math.lsa.umich.edu/∼fomin/papers.html.

[14] S. Fomin and A. Zelevinsky, Cluster Algebras I: Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[15] S. Fomin and A. Zelevinsky, Cluster Algebras II: Finite type classification. Invent. Math. 154 (2003), no. 1, 63–121.

[16] C. Geiß, B. Leclerc and J. Schröer, Generic Bases for Cluster Algebras and the Chamber Ansatz, preprint, arXiv:1004.2781.

[17] C. Geiß, B. Leclerc and J. Schröer, Kac-Moody Groups and Cluster Algebras, preprint, arXiv:1001.3545.

[18] C. Geiß, B. Leclerc and J. Schröer, Cluster Structures on Quantum Coordinate Rings, preprint, arXiv:1104.0531

[19] M. Gekhtman, M. Shapiro and A. Vainshtein, Cluster Algebras and Poisson Geometry, Mosc. Math. J. 3 (2003), no. 3, 899–934.

[20] M. Gekhtman, M. Shapiro and A. Vainshtein, Cluster algebras and Poisson Geometry, Math. Surveys and Monographs, AMS, Providence, RI, 2010.

[21] K. Goodearl, Semiclassical Limits of Quantized Coordinate Rings, Advances in Ring Theory (D. V. Huynh and S. Lopez-Permouth, Eds.), Basel (2009) Birkäuser, pp. 165–204.

[22] K. Goodearl, A Dixmier-Moeglin Equivalence for Poisson Algebras with Torus Actions, in: Algebra and Its Applications, Contemp. Math. 419 (2006) 131–154.

[23] K. R. Goodearl, S. Launois and T. H. Lenagan, Totally nonnegative cells and matrix Poisson varieties, Advances in Mathematics 226 (2011) 779–826.

[24] K. R. Goodearl, S. Launois and T. H. Lenagan, Torus-invariant Prime Ideals in Quantum Matrices, Totally Non-negative Cells and Symplectic Leaves, Mathematische Zeitschrift, doi:10.1007/s00209-010-0714-5.

[25] K. R. Goodearl, E. S. Letzter, The Dixmier-Moeglin Equivalence in Quantum matrices and Quantum Weyl algebras, Trans. Amer. Math. Soc. 352 (2000), no.3,1381–1403.

[26] K.R. Goodearl and M. Yakimov, Poisson Structures on Affine Spaces and Flag Varieties. II. Trans. Amer. Math. Soc. 361 (2009), no. 11, 5753–5780.

[27] J.E.. Grabowski and Stéphane Launois, Quantum Cluster Algebra Structures on Quantum Grassmannians and Their Quantum Schubert Cells: the Finite Type Cases, preprint arXiv:0912.4397v1.

[28] J.E.. Grabowski, Examples of Quantum Cluster Algebras Associated to Partial Flag Varieties, J. of Pure and Applied Algebra, 215 (2011), 1582-1595.

[29] T. Hodges and T. Levasseur, Primitive Ideals of $\mathbb{C}_q[SL_3]$, Comm. Math. Phys. 156 (1993), 581–605.

[30] T. Hodges and T. Levasseur, Primitive Ideals of $\mathbb{C}_q[SL_n]$, J. Algebra 168 (1994), 455–468.

[31] T. Hodges, T. Levasseur and M. Toro, Algebraic Structure of Multi-Parameter Quantum Groups, Adv. in Math. 126 (1997), 52–92. Mathematics 6, AMS, Providence, RI, 1996.

[32] A. Joseph, On the Prime and Primitive Spectra of the Algebra of Functions on a Quantum Group, J. Algebra 169 (1994), 441–511.

[33] A. Joseph, Quantum Groups and Their Primitive Ideals, Springer, Heidelberg, 1995.

[34] B. Keller, On cluster theory and quantum dilogarithm identities, preprint, arXiv:1102.4148.

[35] A. A. Kirillov, Unitary representations of nilpotent Lie algebras, Russian Math. Surveys 17 (1962), 57–110.

[36] A. A. Kirillov, Lectures on the Orbit Method, AMS, Providence, RI, 2004. Int. Math. Res. Not.,

[37] P. Lampe, A Quantum Cluster Algebra of Kronecker Type and the Dual Canonical Basis, Int. Math. Res. Notices (2010) doi: 10.1093/imrn/rnq162

[38] P. Lampe, Quantum cluster algebras of type A and the dual canonical basis, preprint, arXiv:1101.0580v1.

[39] S. Launois, T. Lenagan and L. Rigal, Prime Ideals in the Quantum Grassmannian, Selecta Math. (N.S.) 13 (2008), no. 4, 697–725. 133–145.

[40] G. Musiker, R. Schiffler and L. Williams, Positivity for Cluster Algebras from Surfaces, preprint, arXiv:0906.0748.

[41] J. Scott, Grassmannians and Cluster Algebras, Proc. London Math. Soc. (3) 92 (2006), no. 2, 345–380.

[42] Ya. S. Soibel'man, Irreducible Representations of the Function Algebra on the Quantum Group $SU(n)$ and Schubert Cells, Soviet Math. Dokl. 40 (1990), 256.

[43] Ya. S. Soibel'man, The Algebra of Functions on a Compact Quantum Group, and its Representations, Leningrad Math. J. 2 (1991), 161–178;Correction, (Russian) Algebra i Analiz 2 (1990), 256.

[44] L. L. Vaksman and Ya. S. Soibel'man, Algebra of Function on the Quantum Group $SU(2)$, Func. Anal. Applic. .22(1988) 170–181.

[45] M. Yakimov, Invariant Prime Ideals in Quantizations of Nilpotent Lie Algebras, Proc. London Math. Soc. 101 (2010), 454–476.

[46] M. Yakimov, A Classification of H-primes of Quantum Partial Flag Varieties, preprint,Proc. Amer. Math. Soc. 138 (2010), 1249–1261.

[47] M. Yakimov, Cyclicity of Lusztig's stratification of Grassmannians and Poisson geometry, In: Noncommutative structures in mathematics and physics, Eds: S. Caenepeel, J. Fuchs, S. Gutt, Ch. Schweigert, A. Stolin, and F. Van Oystaeyen, pp. 258–262, Royal Flemish Acad. of Belgium for Sciences and Arts, 2010.

[48] M. Yakimov, Strata of prime ideals of De Concini-Kac-Procesi algebras and Poisson geometry, prepring arXiv:1103.3451.

[49] M. Yakimov, On the Spectra of Quantum Groups, preprint, arXiv:1106.3821.

[50] S. Zwicknagl, Toric Poisson Ideals in Cluster Algebras, preprint, arXiv:1009.2936.